so let me begin by giving a summary of the last lecture let me redraw the circuit

so i have a resistance r i have an inductance l i have a capacitance c all three of them connected to a source of alternating voltage given by vt equal to vm sine omega

so we have seen that what the role resistance played in dc circuits is defined to be impedance which depends on the values of r l and c and is usually denoted by z which is given by r square plus xc minus xl whole square and we had seen that xc is given by 1 over omega c and xl is given by omega l

so this is capacitive reactance and this is inductively attached now notice unlike resistance the impedance of course depends upon the values of r l and c but it also depends upon the frequency of the source and we had seen that corresponding to this source voltage the value of the current i is given by i m sine of omega t plus phi where the amplitude of the current is given by vm over z and phi is the phase by which current leads the voltage now what we did last time is also to look at a graphical interpretation of what is happening suppose i take the direction of the current as this now remember that the direction of the current is the same as the direction of the voltage drop across the resistance

so i will write this direction as vr which is actually equal to i times r which is also along the direction of current now if you recall that an inductor the current lags the voltage another way of saying that is for an inductor voltage leads the current by 90 degree

so therefore if in the same diagram i am drawing the voltage across the inductor i'll draw it like this

so this is let's say vl and for a capacitor since capacitive voltage would lag the current because capacitive current leads the voltage i would put the capacitive voltage along this direction and let me without loss of any generality take the capacitive voltage to be larger than the inductive voltage and of course if the reverse is true then i would my drawing would change accordingly now

so therefore the net voltage due to the inductor and the capacitor because they are aligned oppositely would be obtained if i subtract from vc vl

so this would come let's say here

so this is minus vc minus vl now what i do is this if i complete this parallelogram then this would give me the direction of the source voltage

so let us call this v s for the source and of course the this rectangle would be in the upper half plane if my vl were greater than vc now this angle phi is the angle by which the supply voltage lags the current the angle by which and here since the voltage lags the current the circuit is dominantly capacitive

so let me give a few more examples over what i did last time

so i have a 100 micro farad capacitive in series with a 40 ohm resistance which are connected to a 110 volt this is rms 60 hertz supply the question is what is the time lag between the current maximum and the voltage maximum okay first since it's a 60 hertz supply which is a linear frequency

so 60 hertz corresponds to omega equal to 2 pi times 60 which is approximately equal to 377 radians

so this capacitive reactance is 1 over omega c and that is equal to 1 over 377 and c we have given is 100 micro farad

so that is 10 to minus 4 current and that if you calculate works out to about 26.

5 ohms i can immediately calculate the impedance of the circuit impedance of the circuit is obviously 40 square plus 26.

5 whole square and that if you calculate works out to approximately 48 volts now i have given you the voltage in rms

so i can immediately calculate how much is irons the current

so the rms current is simply 110 divided by 48 which is equal to 2.

29 amperes and this corresponds to a maximum or the peak current which is obtained by multiplying this 2.

29 with square root of 2 and that's equal to 3.

24 now notice that in this case since i have only a capacitor and a resistance that is an rc circuit my current leads the voltage now the angle by which current leads the voltage is given by phi equal to tan inverse of xc over r and that if you put in the numbers and look up a trigonometric table it works out to 0.

58 radians now the time lag between the voltage maximum and the current maximum is obviously given by phi by omega and the reason is the expression for the current is im sine omega t plus 5 whereas the corresponding expression for the voltage is vn sine omega t we know that sine omega t becomes maximum at t is equal to pi by 2 omega whereas sine of omega t plus phi becomes maximum when t is equal to pi by 2 omega minus 5 by omega

so therefore the lag between current maximum and the voltage maximum is given by phi over omega the timeline between current and voltage maxima is phi over omega and that's equal to 1.

55 milliseconds now let's see what happens if i increase this

so if i take the linear frequency f to be equal to 1 kilohertz which corresponds to a value of omega which is given by multiplying this number with 2 pi and that's equal to 6 2 8 3 radian per second and omega times c is 6283 multiplied by 100 micro farad that is 10 to the power minus 4 farads and that's equal to 0 point approximately 0.

63 omega and correspondingly 1 over omega c which is equal to your xc this works out to just 1.

59 ohms i can calculate the impedance z because the resistance is 40

so 40 square plus 1.

59 square square root and that works out to about 40.

03 ohms now look at what is my xc by r remember my xc is small 1.

59

so this 1.

59 divided by 40 is equal to 0.

039 and correspondingly the tan inverse of this since this number is rather small the phase 5 is tan inverse of xc by r which also works out to approximately 0.

03 radius and this time lag that you get now is given by 0.

039 that is 5 divided by 6 to 8 3 and that's equal to 6.

3 into 10 to minus 6 seconds

so you notice the current is gradually becoming almost in phase with the voltage in other words as i increase the frequency of supply the capacitor is becoming more and more conductive remember for a dc the capacitor was an open circuit and did not allow the current to pass

so with increasing frequency the capacitors turn out to be more conductive

so last time we did the graphical analysis of the lcr circuit

so let me now take the mathematical analysis and that goes as follows i can write down it is the analytical solution

so using kirchhoff's law the loop law i can write down the equation as ldi by dt plus ir plus q over c is equal to v m sine omega t where q by c comes because that is the voltage drop across the capacitor now this equation can be converted into a second order differential equation either in charge or in current

so observing that i is equal to dq by dt you can do that i decide since i am only interested in current for the moment let me differentiate this equation once more

so we get l d square i over dt square plus rdi by dt plus 1 over c times dq by dt which is equal to i that's equal to vm omega cosine of omega t

so look at this equation the right hand side is a trigonometric function and on the left hand side i have got current i and its differentiations once or twice with respect to time

so what i do is i assume a solution of this form i is equal to i m sine of omega t plus phi where phi as i have explained several times is the phase by which the current leads the voltage and if you differentiate it once you get di by dt equal to im omega cosine of omega t plus phi and a second differentiation gives me d square i over dt square equal to minus i m omega square sine of omega t plus phi minus because the differentiation of cosine gives me minus sign and if you substitute these things into this equation that we have i get i m omega i get minus l omega plus 1 over omega c times sine of omega t plus phi then plus r cosine of omega t plus phi that is equal to vm omega cosine of omega t and of course we need to determine uh what are these quantities i m and phi

so what i get from here is im omega minus l omega plus 1 over omega c times sine of omega t plus 5 plus r cos omega t plus 5 and that's equal to vm omega cosine of omega t observe that this quantity here is xc minus x i can simplify the left hand side of this expression by taking r equal to some a cosine theta and xc minus xl to be equal to a sine theta where of course i need to determine what are a and theta but you can immediately see from these two relationship i have a square equal to a square cos square theta plus a square sine square theta which is r square plus xc minus xl whole square and that if you recognize is nothing but the impedance square z square which tells me that a is simply equal to z and tangent of theta is obtained by dividing the second by the first is xc minus xl divided by now with this identification and canceling the common terms omega from both sides of this equation i get i m z cosine of omega t plus phi minus theta that is just because i have taken this r to be a cos theta

so this is a cos theta cos omega t plus five this plus a sine theta omega t plus five

so this is what i get and that quantity is equal to vm cosine of omega now if you compare the two sides of this expression what i get is i m times z is equal to vm which tells me that the maximum current of the current amplitude is given by vm divided by z and this theta is simply equal to phi

so my solution is given by i equal to i m which is vm over z sine of omega t plus phi with tangent of phi which is now have been shown to be equal to tangent of theta equal to xc minus xl divided by let us look at a an interesting property of lcr circuit and this is known as the resonance the phenomena of resonance you have come across even in mechanical circuits for example a driven pendulum now we know that when the driving frequency equals the natural frequency of the problem then the amplitude rises substantially now this phenomena is called the resonance

so let me look at what are the properties of this resonance now

so i maximum is vm divided by square root of r square plus xc minus xl whole square where xc is 1 over omega c and xl is omega times l now notice one thing that the impedance is minimum impedance is what is there in the denominator

so impedance is minimum implying thereby that the current is maximum when xc becomes equal to xl

so when 1 over omega c is equal to omega l that implies that when omega is equal to omega 0 equal to square root of 1 over lc this omega 0 is my definition of a resonant frequency notice that the resonant frequency does not depend upon

dissipative element like resistance it is entirely decided by what the values of l and c are and at the resonance my current becomes maximum and its value becomes vm divided by r and the phase remember we said that the tan phi is xc minus xl divided by r phase five becomes equal to zero phase five becoming equal to zero means that the current is in phase with the supply
 so let's look at what it implies let me plot the current against the impressed frequency now i will do it for different values of r now what you find is this that as you increase the frequency then your im at follows a curve like this
 so this for instance is for a particular value of r
 so let us call it r1 now suppose i now decrease the value of r making it let's say r2 then what will happen is that it will become sharper and the maximum current will be more and this frequency at which
 so this is this was r1 this is r2 which is less than r1
 so this is the frequency is omega 0 and this is of course my current now remember what is actually happening in order to have a resonant frequency i need both l and c must be there and this is because if you recall the two reactances are aligned oppositely and
 so therefore there is a cancellation which is possible between the xl and xe the cancellation becomes exact at the resonant frequency and the maximum current rises now this instantly is the principle by which various tuners work for example when you tune in a particular radio station now what you find is if you are going on rotating the dial when your capacitive frequency of the circuit that is inside the radio tuner matches the natural frequency in which the signal was coming in now that is the time when you receive the signal sharply
 so this is what is used in radio tuning and any many other tunings now having defined what is resonance let me try to look at the sharpness of resonance but before that let us look at that suppose i am plotting the impedance z what type of situation they have now remember at the resonant frequency the impedance is minimum on either side of the frequency the impedance rises whether the it is a capacitor reactance is more or the inductance reactance is more that is totally material but look at what we are having
 so what we get is something like this supposing this corresponds to i'm plotting it against omega supposing this corresponds to omega 0 omega of frequency doesn't matter and
 so what happens is in the capacitive section i get this and for inductive part
 so this is xl greater than xc this is xc greater than xl now notice if omega is greater than omega alternatively if omega square is greater than omega 0 square then what we are saying is omega square is greater than 1 over lc
 so that i have l omega greater than 1 over omega c and l omega if you recall is the inductive reactance
 so this is what we are talking about in this part and reverse is of course true if i have omega square less than omega 0 square now what is the idea of this resonance now since the impedance is minimum and you remember the current is proportional to 1 over z because current is basically maximum current is vm by z then the current is maximum and the power which goes as i square z if z is minimum current is maximum the an i square z which is the power absorbed by the circuit is also maximum at omega equal to omega z now there is a prescription for deciding how to measure the sharpness of these resonances
 so what we do is this that when we increase or decrease the frequency from omega 0 starting with omega 0 i can increase the frequency or decrease the frequency then i look at that point where the power absorbed is half the maximum power absorbed
 so let us look at that picture again
 so this is my current and this was the resonance this is the way the im varies

with omega let me call it i m because i am still talking about maximum current and and this is the value of i m and this is actually the maximum as a function of omega now

so what we do is this that

so this is where my current is maximum

so let me say this is equal to omega 0 and this value let me just write it as i m maximum

so this maximum is as a function of omega now what do you do is this we increase the frequency starting from omega 0 or decrease the frequency on either side and where we have the power average power absorbed is half the maximum value okay

so increase or decrease frequencies starting with omega 0 till power absorbed average of course is half the maximum power absorbed this is what happens at omega z now remember that my power expression was at this point because z essentially is r

so i had i square r that was the power now i want this power to be 50

so 0.

5 times that

so these are called half power point which i can rewrite as i by square root of 2 whole square times r

so basically i am looking at the points where the maximum current has dropped to about 70 percent of the value that is one over square root of two

so this is these are the two points if you like we'll call this omega one is the upper half power point and omega 2 as the lower half power point now this width here

so omega 1 minus omega 2 or alternatively if you are looking in the language of the frequency then correspondingly dividing it by 2 pi that's called the bandwidth let us write this as delta omega

so this is bandwidth expressed in radian frequency

so you could also write it as f1 minus f2 which will be in hertz

so this is bandwidth usually bandwidth is just written as bw

so this is delta actually it's 2 times omega because the if this is delta omega

so this is also another delta omega

so this is my definition of the bandwidth

so let me now do a little more quantitative calculation

so let us suppose omega 1 is equal to omega 0 plus delta omega you have said this is a higher half power point and omega 2 equal to omega 0 minus delta

so i said that 2 times delta omega is my definition of bandwidth

so what it means is this if you have a sharper resonance then your bandwidth becomes smaller because the then the peak is much sharper

so sharper resonance implies lower bandwidth now let us look at the point omega 1

so at omega 1 i have got

so omega equal to omega 1 i have got i m equal to vm divided by square root of r square plus omega 1 l minus 1 over omega 1 c whole square and this is by our definition of the half power point is i m max divided by square root of 2

so which is equal to since i am max is nothing but v m by r

so it is vm r into square root of 2 in the denominator

so basically what i have is this by squaring both and doing some algebra i get the following i get r square plus omega 1 l minus 1 over omega 1 c whole square is equal to 2 r square and

so that tells me that omega 1 l minus 1 over omega 1 c is equal to r now of course if the the solution of this equation is plus or minus r it would depend upon which one to take depending upon whether the inductive reactance is greater

or the capacitance is greater but i already know that omega 1 is omega 0 plus delta omega
 so this times l minus 1 over omega 0 plus delta omega times c is equal to r
 so let us look at this
 so let me take omega 0 l common i am left with 1 plus delta omega by omega 0 minus let's again take omega 0 c common here i have got 1 plus delta omega by omega 0 in the denominator and if i take it to the numerator using a binomial i get 1 minus delta omega by omega 0 assuming delta omega is small and that's equal to now notice that by definition omega 0 l is equal to 1 over omega 0 c because omega 0 is the resonant frequency
 so therefore when you open up this that term will cancel out and i will be left with l times delta omega from this term plus plus because there is a minus here minus here 1 over omega 0 c times delta omega by omega 0 that's equal to r and this term is nothing but omega 0 l at the resonance frequency
 so therefore both these terms are l times delta omega
 so i'm left with omega 0 l times 2 delta omega is equal to r which tells me that delta omega is equal to r divided by 2 l sorry there is no omega 0 here because omega 0 cancels with this omega 0 we define the sharpness of the circuit through a quantity called the quality factor
 so quality factor of a circuit is omega 0 by 2 delta omega which is equal to omega 0 l divided by r and that is represented by q
 so let me give a few examples i have a series lcr circuit which has a resonance frequency at let's say one kilohertz
 so in a series lcr circuit resonance frequency omega0 actually i am giving f0 is equal to 1 kilohertz and quality factor is given to be 100 now suppose i double r l and c are doubled each of them is doubled what would happen to q now look at this i know omega 0 is 1 over square root of lc
 so if you double both l and c then the denominator here will become increased by a factor of
 so that omega 0 reduces by a factor of 2 if l and c are doubled the but remember my q is omega 0 l divided by r
 so we have seen that omega 0 would reduce by a factor of 2 but since you are increasing both l and r nothing would happen to this factor
 so therefore q will become 50.
 let me give you another example let me take an lcr circuit with the alternating voltage being given by 240 volts times sine omega t i have been given l is equal to 10 milli henry c is equal to 1 micro farad and r is equal to 40 ohms
 so let us find various data connected with the this problem first is let us look at what is resonant frequencies resonance frequency of the circuit is given by omega 0 equal to 1 over square root of lc
 so l is given to be 10 milli henry
 so that it is 10 to the square root of 10 to the power minus 2 henry c is 1 micro farad meaning there by 10 to the power minus 6.

 so taking square root it is 10 to the power 4 radians per second what is the amplitude of the current at resonance
 so at resonance the im itself is maximum that is the amplitude is maximum and
 so i am max is given by vm divided by r that is equal to 240 divided by r that's equal to 6 amperes and the quality factor is given by omega 0 l divided by r here r was of course 40 ohms and
 so that's equal to 10 to the power 4 there l is 10 power minus 2 divided by 40
 so that's equal to 2.
5 what is the voltage across the inductor at resonance
 so v l max

so that's obviously equal to i m max times omega 0 l that's your reactance which is 6 multiplied by omega 0 is 10 to the power 4 and l is 10 to the power -2 henry

so that's equal to 600 volts let me give another example let us consider a circuit for which the inductance i take it a rather large inductance 300 capacitance is 27 micro farad and resistance r is 7.

4 ohms what should be done if you want to improve the sharpness of resonance by reducing the full width at half maximum fwhm by a factor of 2.

so the question is what should we do to reduce fwhm by a factor of 2.

well remember that omega 0 is 1 over square root of lc which is 1 over now l is 300 c is 27 into 10 to minus 6.

so this is 1 over 9 in the denominator and 10 to the power 3 in the numerator

so it is 111 radian per second and the quality factor is omega 0 l over r which is 111 multiplied by 3 divided by 7.

4 and that's approximately equal to 45 now suppose keeping omega 0 fixed i want to reduce delta omega

so if i recall that the expression for q is omega 0 l by r one of the possibilities would be to reduce r because i want to double q which is the same as doubling ah reducing the width of the full width at half maximum by a factor of two

so one possibility is of course to reduce r which is resistance or of course equally to increase l but technically it turns out that manipulating inductance is a lot more difficult thing in fact what is done in lcr circuits is to use capacitors and resistances to be adjusted that is because resistances can be varied by using variable resistances and also it is possible to change capacitances but manipulating l is lot more difficult and hence the best that you can do is to reduce r by a factor of two

so the solution would be to reduce r by a factor of two

so let me quickly summarize what we did today earlier we had seen how to describe lcr circuit by graphical methods

so today we actually solved a second order differential equation and solved the circuit for current and obtained the analytical solution getting expressions for both the current maximum and the phase lag that the current has with respect to the supply voltage having done that we defined a property of lcr circuit known as resonance and what happens is that if you are able to control the frequency of the impressed voltage then at a particular frequency which is given by 1 over square root of lc the current amplitude rises substantially and and that frequency is known as the resonant frequency the property of the resonant frequency is that that is the frequency at which the circuit absorbs maximum power from the source we defined the sharpness of resonance by finding out what are the half power points on both sides that is what should be the current value current amplitude values for which the power absorbed by the circuit is half the maximum power it could absorb at omega and these two symmetrically located points where the current maximum is the power maximum is half is what is known as full width at half maximum and smaller this width is sharper is the resonance and with that in mind we defined what is known as the quality factor of a resonant circuit you