

today we will proceed further there is another important concept which needs to be addressed with respect to rotation about a fixed axis that is namely the angular momentum

so so we will have this angular momentum today's topic the case of rotation about a fixed axis so what are the various things we propose to study today we will first arrive at an expression for the orbital angular momentum angular momentum this is for a symmetric body rotating about a fixed axis then we will see that how the principle of conservation of angular momentum uh looks like and we will consider a couple of examples then third thing is rotation rolling and slipping these are the various kinds of motions possible for rigid body we will focus our attention on rotation and rolling for slipping maybe we will spend a time later and we will consider an example this is what we propose to do

so i have here diagram and a rigid body is rotating this kind of diagram we have seen and this is the center this is the radius vector this is with reference to a suitable origin woh this is the point p the op vector is what we call it as the position vector r okay and the particle is going round like this

so i can indicate it by this arrow this is the z axis i have x axis here x y z here okay

so um ω is the angular velocity let me say and then m is the mass of the particle and then here we have let us say two things one is the linear velocity v m is the mass of the particle linear velocity okay and angular this angular velocity vector also i need to indicate remember this vector is tangential to the circle

so it is not vertically upwards even though it looks like this in the diagram now uh the body is rotating no therefore it has got an angular velocity of ω and i need to draw a vector here which is the ω vector and this is these two vectors are actually parallel we will see which is not clear here in the diagram manner

so we study the angular momentum in the special case of rotation about a fixed axis

so what is the general expression the general expression is utilial for a single particle is r crossed with p or crossed with p

so i have r is equal to o p this case r is equal to ot vector that is equal to o p vector is oc plus c p oc plus cp and where the p is equal to the momentum is what this is momentum vector is m times the velocity vector linear velocity these things are fairly standard okay now i will calculate l angular momentum vector is r is oc plus c p cross this with m times v i need to show the cross sorry this is equal to oc crossed with m times v plus c p crossed with m times v remember the cross product is distributive and now cp has got a name this is what is called as r perpendicular this we have seen it in earlier lecture

so v will be value of v would be velocity linear velocity would here would be r perpendicular times ω therefore i can write now l is equal to oc crossed with m ω m v sorry plus plus cps are perpendicular are perpendicular squared then i have m then ω what's the direction that is the k direction we need vector k along this

so i have here this is equal to i will write this as this is the expression for z component of angular momentum

so i will write this l as l sub z plus oc times mv now l z is parallel to the fixed axis now this direction of uh this direction k can be obtained from this right hand rule also that's how we can get it for that matter now what we have is l z um l is it parallel um elizabeth is parallel to the fixed

axis k but you can't say that l itself is parallel to the z -axis. I can't say that this is parallel to the vector \hat{k} . This is not correct. This is right. This is wrong. Okay, in general, the object angular momentum is not along the axis of rotation. l and ω need not be parallel. In general, l and ω need not be parallel. But in the case of an object rotating about a z -axis, the two for symmetric bodies l and ω are parallel. Now we will calculate what is total angular momentum. This whole body is made up of

so many masses therefore total angular momentum is sum of all the angular momentum because angular momentum is a vector quantity the summation runs over i corresponding to all particles

so this is l_z is z components of all particles plus sum over i r_i crossed with m times v_i . It is straightforward generalization. Now we call this quantity l_z is the cumulative sum of all the z components. This term plus the other term is l_{\perp} . This is the term

so l_{\perp} is the other other quantity which is n circle

so what is l_z ? Let me write this l_z vector. l_z is equal to summation over i r_i perpendicular i squared times ω_k . ω is the same for all at mass at every point at every point therefore ω can be taken out

so l_z is equal to what is this quantity. This is the moment of inertia. This is the moment of inertia of of the body about this particular axis of rotation. This is the r_{\perp} . Let me indicate here. This is the r_{\perp} perpendicular at this particular point. There is a mass m_i and its perpendicular distance from here and the distance from C is r_{\perp} . Therefore this is the i ω times k

so this equation is something reminds us a bell should ring in our mind it is something similar to $p = mv$ is equal to mv the case of linear motion

so in the case of symmetric rigid bodies what happens is for every for a symmetric rigid body for every i r_i for every i for a given i for every particle which has a velocity v_i there is another particle with a velocity $-v_i$ if it is a symmetric body given this r_i if there is going to be a velocity in this direction there is going to be another particle which is diametrically opposite at the same distance with the velocity $-v_i$ therefore these two components will cancel

so you are left with that l_{\perp} is equal to zero

so hence we have for a symmetric rigid body rotating about a symmetry axis l_z is equal to $I \omega$ times k

so the axis of rotation is also same as the direction of ω . Now for objects which are not symmetric about the axis of rotation l is not equal to l_z . That that you must keep in mind and helps does not lie along the axis of rotation. In such cases we will consider a few illustrations. Example one let us say that I have a circular disk. I have a circular disc. It is a circular disc. This is the axis of rotation. This axis of rotation this is z . ω and its radius is r . Now I want to write down what is the object what is the angular momentum vector. Angular momentum vector is equal to yeah this is symmetric body this is the the axis is also symmetric symmetry axis

so therefore l is nothing but $I \omega$ times k . Fine now what is the moment of inertia of a circular disc yesterday we had seen $\frac{1}{2} m r^2$ and angular velocity is ω this times unit vector which is along the z direction. Now I can do a slightly different problem. Example one here this is example two what I'll do is in the earlier problem what I have done is I have we have coincided the z axis with the symmetry axis of the body. In fact suppose the

z axis lies outside and we have the same situation same body okay everything is same it is rotating with ω and these two axis are parallel these two axis are parallel then again l will be equal to now this symmetric rigid body rotating about the symmetry axis therefore its angular momentum is given by $I \omega$ times k but only thing is different this I is moment of inertia about the symmetry axis that is $m r^2$ by 2 but this is we are calculating with respect to this therefore we want to know the moment of inertia about the symmetry about the z axis therefore this is $m d^2$ squared this is what we call it as parallel axis theorem this is the parallel axis theorem this I have indicated as parallel axis theorem this times ω times k that's demand

so we have got the moment of inertia of this object we will make few comments before we proceed further

so l is equal to $I \omega$ times k now what is dl by dt dl by dt is equal to I into $d\omega$ by dt times k this is equal to $d\omega$ by dt is α therefore $I \alpha$ times k great vector $I \alpha$ study we have seen is nothing but the torque okay

so

now we calculate since l is equal to l_z plus l perpendicular

so what do we get $d l_z$ by $d t$ is equal to τ times k and $d l$ perpendicular by dt is equal to zero now this gives rise to the principle of conservation of angular momentum principle of conservation of angular momentum well this is generally $p c$ am some observation the total angular momentum of a system is constant if the resultant external torque acting on the system is zero there the total angular momentum of a system of a system is constant in other words it is concerned if the resultant external torque acting on the system system is zero now we are considering with respect to symmetric rigid bodies therefore we have the angular momentum initial angular momentum is same as final angular momentum times ω is equal to constant this is a sort of statement for the angular momentum conservation okay now this conservation of angular momentum is something similar to it should again a bell should ring principle of conservation of linear momentum the case of linear motion just for comparison purposes i am indicating we will do an illustration of this we will do a problem or illustration now say i have a situation is like this i have a i have a cylinder i have a cylinder this is the axis of the cylinder is the axis of the cylinder okay this is the horizontal axis axis of the cylinder this is horizontal this is horizontal uh there is a mass there is a bullet which comes and hits it actually the way i have indicated it looks like it is a normal this comes m and v naught okay so the bullet hits the uh the direction of the bullet is perpendicular to the horizontal axis of it means at a particular distance let us say the distance between these two is d the bullet hits the cylinder at a particular distance d from the axis okay and r is the radius of the cylinder

so see i need to say that this is the line of motion the line of motion of the bullet it is perpendicular to the axis of the cylinder even though in the figure it may not be like that that's why i am writing this okay now we uh various things can be calculated at least we can calculate the angular speed of the system after the projectile strikes and gets embedded on the cylinder initially the cylinder is at rest after the bullet strikes the cylinder the the whole system will begin to rotate we can calculate the angular speed of the entire system

so here we can apply the principle of conservation of angular momentum because there are no external torques so before collision before collision

so before collision only the bullet has the angular momentum with respect to the axis of the cylinder and its value is $mvnaught d$ okay after that after the collision what is its angular momentum its angular momentum is i times ω total angular is i times ω what is i it is nothing but i of the solid cylinder plus i of the projectile because it the projectile got itself embedded onto the cylinder this times ω okay i will call it as i final therefore now we can equate this to

so i have $m r^2$ by two solid cylinder is $m r^2$ by two is the moment of inertia plus after it got embedded on the surface the mass of the bullet is m as we said it is a distance of r it got embedded on the surface $m r^2$ times ω this is equal to initial angular momentum initial angular momentum is $m v naught d$ that corresponding to only that of the bullet therefore this implies ω is equal to $m v naught d$ divided by $m r^2$ plus $i r^2$ in fact this expression can be made use of to find the velocities of a bullet because the bullet will whisk fast very fast

so once you make this bullet hit a cylinder get it embedded on the surface that is important it has to get embedded on the surface then we can measure the ω through that we can measure the value of $v naught$ ok we will consider one more illustration one more example ah the situation is like this i have a circular disc ok this circular disc it has got an axis and it is pivoted

so the disc can rotate about this axis both this side it is pivoted so this is and the mass of the entire disc is m and r is the radius the center we will call it as c now as it is rotating with a constant angular velocity ω here we have a mass m and it begins to roll towards the center and reaches a particular point let us say c such that oc is equal to x is it moves along what we are the question is there is a point c it has to cross

so we want you to calculate the ω

so calculate ω calculate ω question what is calculate ω when it is when the little mass reaches c that is the thing initial ω initial value of angular velocity is ω as the this m is uh considerably heavy it is not negligible compared to m therefore as this mass moves towards woh the whole angular velocity will change now let us see what happens

so this is a circular platform i will call it as cp circular platform let me repeat the problem there is a circular platform of a mass m of radius r which is pivoted about a particular point o which is rotating about an axis with angular velocity constant angular velocity ω a mass small m it begins to move towards the center initially it is at the rim of this circular platform you are required to calculate the angular velocity of the entire object when it reaches c that is the question now ah again there are no external torques therefore the angular momentum orbital angular momentum sorry initial is equal to orbital angular momentum later angular motion later at some instant of time now first let us calculate what is the we need to make use of the formula L is equal to i times ω right

so we need to know what is the initial moment of inertia of this system initial moment of the system is ie of cp plus i of mass this is equal to the circular disk therefore $m r^2$ by 2 cp is the circular platform plus

little m into r^2 in the beginning now i is equal to again the same thing this is equal to $m r^2 \omega^2$ but now the mass is at the point c which is a distance x therefore $m x^2 \omega^2$ now I am going to make use of the principle of conservation of angular momentum which says moment of inertia initial times ω_i is equal to moment of inertia times later ω_f right

so we equate these two and we can uh

so $m r^2 \omega_i^2 + m x^2 \omega_f^2 = m r^2 \omega_c^2 + m x^2 \omega_c^2$ therefore ω_c when it ω_c at c means the angular velocity entire system when the mass is at c okay is equal to $m r^2 \omega_i^2 + m x^2 \omega_f^2$ divided by $m r^2 + m x^2$ ok now it is clear that ω_c is equal to this ω_c is greater than ω_i that is because look at this numerator and denominator the denominator you are having a smaller quantity x^2 here therefore ω_c is going to be greater than ω_i what does it mean it implies that the rotational kinetic energy at c is greater than rotational kinetic energy initially that means as the little mass m moves towards the center the kinetic energy of the entire system is increasing how does this happen it happens because it happens because now if this as the mass m moves towards bow in order to keep itself in a position it should apply there should be a centripetal force it has to do work to create centripetal force therefore energy is given to the system the kinetic energy is the same energy is given to the system therefore the kinetic energy of the system increases one can calculate what is the amount what is the increase in kinetic energy this can be calculated because the expression is $\frac{1}{2} I \omega^2$ we know what is the kinetic energy initially $\frac{1}{2} I \omega_i^2$ we are calculating what is ω_c therefore again we can calculate the kinetic energy calculate the difference right actually this work done as this body moves along is transferred to the internal energy of the system right now we will move on to the next topic next topic is next topic is rolling I will call it as rotation rolling and slipping now I will give a little bit of motivation for this the motivation is like this I have a table top let us say what I do is I have a disc which is rotating about its axis with some angular momentum ω I am having a disc which is rotating about an axis with angular momentum ω and I place it gently the disc is the rotating disc is placed gently the rotating disc is placed gently on the table let us say it is a perfectly frictionless frictionless table right now I will consider this point as a this point as b this is I will consider some point here which is c from the center okay

so let us say that oc is equal to r by 2 what is happening what is the linear velocity at a linear velocity at a is $r \omega$ and radius is r therefore $r \omega$ what is the linear velocity at b linear velocity at b is equal to $r \omega$ what is the linear velocity at c again whatever is the radius radius is r by 2 and ω remains the same why are we giving this example sir it is just to show that because the table is frictionless and the disc is rotating with an angular velocity if it is placed on it it is placed on the the rotating disc is vertically placed on the table very gently very gently means that no push or anything slipping no push or anything then what happens is that when you calculate the linear velocities at various points these are the values okay now the question is it will the disk will only rotate disk one disk only rotates now the question is will it roll no it will not roll it will not roll if you place a rotating disc vertically on a perfectly frictionless table the disc will not roll

this is the point which i want to stress here and okay now we will consider uh rolling motion what is actually a rolling motion a rolling motion is a disc will rotate about an axis and also it will move forward something like uh when you cycle or any two wheeler the wheels will rotate about the axis and also the wheels will move forward

so there is a translational motion as well as rotational motion now i will draw the axis this point i will call it as p_1 this i will consider as p naught this is center i will call it as c okay now the whole thing is rotating angular velocity is ω not

so at this particular point what is the linear velocity linear velocity is v_1 right right here it will be in this direction now suppose i take any point here let us say i will take a point here first thing is this this will have a center of mass this particular point will have a velocity v_{cm} at the center it will move because it is rolling as well as rotating there is a translational motion therefore the center of mass will have a velocity which i will call it as v_{cm} i am not writing the vector

so that it will get cluttered but otherwise its a vector quantity direction is indicated here when i take a particular point here what happens and now to find the velocity here i should join these two ok this particular point will have a center of mass this will have a same v_{cm} right and now what i should do is i should there is going to be a this this i will call it as r then this is how it will have i will call it as this quantity will be the linear velocity this linear velocity at p is the linear velocity vector so the net resultant is going to be i need to compound these two right which i am not indicating it over here

so if i want i can do it here this this this portion alone i amplified here this is v_{cm} i'm magnifying it rather and then this is v_p linear velocity so i can i can complete this is what is going to be the actual velocity at this particular point okay this is p i am magnifying this portion alone here right at p naught at v_p naught at p naught at p naught what will happen due to rotation it is exactly same as v_p naught but uh

so v_p naught at this particular point its linear velocity must be same as p center of mass in other words when the center of mass motion is like this here it will have motion and then its linear velocity will be here both of them should be equal this is what is equal to $r\omega$ naught

so there is no at this particular point p naught when it is rolling it should be at instantaneous rest that is what you call the p naught is at instantaneous rest why it is at instantaneous rest that is because its linear velocity should match the velocity of the center of mass okay

so we call it as v_{cm} velocity of the center of mass should be same as $r\omega$ not if this happens as long as this is maintained this is the condition for rolling without slipping condition for rolling without sleeping

okay now what about the instantaneous what about at p_1 p at p_1 is equal to velocity of center of mass plus $r\omega$ therefore this will be equal to 2 times v_{cm} this is again for rolling right okay it will have a center of mass velocity as well as the linear velocity and then the linear velocity is same as $r\omega$ naught therefore it is twice v_{cm} now we will derive an expression for the kinetic energy of a rolling motion

so kinetic energy of rolling motion k is equal to kinetic energy of a rolling body of a rolling body is

so remember a rolling body has kinetic energy of translation plus kinetic energy of rotation see you peop students should clearly distinguish between rotation about an axis the translation about an axis both put together is what is known as the rolling motion of a body right

so earlier we had seen ah now we want to recall i recollect something therefore i will draw in a different colored layer we are seeing a kinetic energy of a system of particles i think it is in lecture 2 i guess as soon as we introduce the center of mass this we did as in fact we did a two body problem so kinetic energy of a system of particles is equal to kinetic energy of the center of mass plus kinetic energy of rotational motion about the center of mass that is important this we had done so in the same way the same way we have so kiki the kinetic energy is equal to of what we are the rolling body is equal to first translational motion if the mass is m $c m$ square plus the kinetic energy of the rotational motion about the center of mass this is half i ω square right and moment of inertia is also written in terms of $m k$ square let me use a little $m k$ square where k is the radius of gyration okay which we had seen earlier now k is equal to half $m k$ squared $m k$ squared $v c m$ squared by r squared well how do i write it that is because p of centre of mass is equal to $r \omega$ condition for rolling so this plus translational motion energy for the kinetic energy of transformation motion v of $c m$ square therefore k is equal to half little $m b c m$ squared 1 plus k square by r square this is a very standard formula okay it's a very standard formula so what is that we have done the kinetic energy of a rolling body so we have made use of what you have done is the kge of of a rolling body flowing body is equal to kinetic energy of translation plus kinetic energy of rotation okay this is something similar to this is nothing but what we already have done in the case of many particles both are actually the same we have got an expression for the kinetic energy now we can make use of this expression time request now we can make use of this expression to do a simple problem it is like this what we have is i have an inclined plane i have an inclined plane i have an object it may be sphere or cylinder or circular disc it begins to roll it rolls down so i have a ring and a solid cylinder and a sphere okay now at this point this whatever it is if it is let us say it is a ring or a solid cylinder sphere the object will have only potential energy when it comes here it will have only kinetic energy therefore mgh is equal to expression for kinetic energy is mv squared by 2 v is the center of mass of course 1 plus k squared by r squared we have just derived it right suppose it is a now we will have a small table even if it is little more this object first i will have a circular ring what is its k value radius of gyration is a circular ring or a disc it is only sorry circular ring is r so i will put this expression here and calculate what is v i will get from this implies v is equal to $2 g h$ by 1 plus k squared by r squared square root so this will be gh because k is equal to r so 2 and 2 will get cancelled this is what we will have for circular range the case of a circular disk in case of a circular disk this is um this is r by root 2 therefore it will have 4 by 3 it has higher value than this next we have a sphere solid sphere it is root 2 by $5 r$ radius of gyration is root 2 by square root of 2 by 5 times r then it will be 10 by $g h$ so you realize that uh even though all these objects ring or a solid cylinder sphere have the same radius are same all of them they have the same radius you will find that the solid sphere when it comes to the bottom it will have the maximum it will have the greatest velocity greatest velocity is for the solid sphere therefore greatest kinetic energy you