

welcome back to this lecture of sequence and series we shall start with a quick recap of our development

So far in the previous lectures we introduced the notion of sequence informally by a sequence we mean an ordered list of numbers to be more formal a sequence is defined as a function f from s to r where s is a subset of the set of non-negative integers we saw the notation used for a sequence a sequence can be denoted using the notation a_n n is equal to 1 to infinity a_n are called terms of the sequence and n th term a_n can be viewed as output of the function f evaluated at n that is a_n is actually f evaluated at n where f is the underlying function associated with the sequence a_n n is equal to 1 to infinity other possible notation for sequence is $\{a_n\}$ or we can list a sequence as follows it is advisable to describe the n th term of the sequence namely a_n should also remark that though we have written a_n n is equal to 1 to infinity as in the formal definition of the sequence n may vary from 0 to infinity or it may vary over some subset of the set $n \cup \{0\}$. that is a sequence may start for instance from f_4 onwards we also saw that a sequence can be described in two ways one is called closed form expression were sequence a_n is described by giving a formula for n th term a_n for instance a_n is equal to $1/n^2$ for every n greater than or equal to 1 another way of describing a sequence is recurrence relation or using a recursive formula where instead of writing the n th term a_n in terms of n we write n th term in terms of some of its previous terms we saw a famous example namely fibonacci sequence though we use the notation a_n n is equal to 1 to infinity for a sequence the distinction between a sequence and a set decides to be stressed note that in a set order of elements is not important whereas in a sequence order matters a lot secondly in a set repetition of the elements are usually avoided whereas in a sequence the elements can repeat we have given plenty of example to illustrate all these things however keep in mind that a sequence is distinct from a set lastly though not rigorously we introduce the concept of convergence in convergence we investigate the behavior of a sequence as term progresses in other words we observe what happens to the terms of the sequence as n becomes larger and larger recall that by the notation $\lim_{n \rightarrow \infty} a_n = l$ what we mean is that the terms of the sequence namely a_n approaches closer and closer to the fixed value l as n becomes large enough for instance we have seen that in the sequence $1/n^2$ n is equal to 1 to infinity to be explicit $1/1^2$ $1/4$ $1/9$ $1/16$ etcetera $1/n^2$ etc we can see that as we progress towards the tail end of the sequence the terms are becoming close to zero hence we write $\lim_{n \rightarrow \infty} 1/n^2 = 0$ we call sequence $1/n^2$ to be convergent and 0 is called as the limit of the sequence $1/n^2$ whereas for some other sequences like say 2^n n is equal to 1 to infinity let us say to be explicit 2^1 2^2 2^3 2^4 and

So on we cannot find a number l

So that as we progress towards the end of the sequence the terms of the sequence become close to this number l such a number l does not exist similar is the case with sequence like n^2 n is equal to 1 to infinity such sequences are said to be divergent keeping these ideas in mind let us proceed i would like to recall the remark i made at the very beginning of the previous lectures namely the following in day-to-day life the terms sequence and series can be used interchangeably both are used to denote succession of events or objects however in mathematics these two terms have distinct meaning and we saw that sequence is used for an ordered list of numbers let us proceed to see what a series means for a mathematician crux of this lecture will be defining a series if you recall finding sum of finite number of real numbers is widely used in mathematics in fact in day-to-day life ranging from a grocery shop for instance to a laboratory where sophisticated experiments are being conducted let us recall that while finding sum of finitely many real values say for instance a_1 a_2 a_3 they are real numbers and we would like to find $a_1 + a_2 + a_3$. note that the order in which we add these terms $a_1 + a_2 + a_3$ does not matter that is to say $a_1 + a_2 + a_3$ is same as $a_2 + a_1 + a_3$ which is same as $a_3 + a_2 + a_1$ and

So on if you recall permutation and combination there are three factorial ways in which we can find sum of three real numbers six different orderings are possible however all these orderings ultimately result in same sum to find sum of finite number of real numbers we can first find sum of two real numbers and to this sum we can add a third real number see what comes out at the end summing up in the process of finding sum of finitely many real values the order in which the terms appear doesn't matter at least mathematically of course when dealing with certain figures we may observe that certain

order is more convenient to work with than certain other orders for instance if we are asked to find sum of say 247 198 and 2. as we observed previously we can add them in 6 different orders however 198 plus 2 equal to 200 which added up to 247 to yield 447 gives the most convenient order

So we can always resort to such shortcuts to find some of finite number of real numbers now consider the question of finding sum of infinitely many real numbers to illustrate let me give you an example let us recall decimal expansion of a real number for instance when we say decimal expansion of $10/3$ is 3.333 etc non-terminating recurrent decimal expansion what do we mean really we mean to say that $3 + 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \dots$ is equal to $10/3$. this is what we mean knowingly or unknowingly in decimal expansion we deal with sum of infinite number of real numbers and a natural question here is the following if we increase number of summands that is instead of dealing with $3 + 3 \times 10^{-1}$ if we deal with $3 + 3 \times 10^{-1} + 3 \times 10^{-2}$ instead of dealing with $3 + 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3}$ suppose we deal with $3 + 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3}$ for terms and

So on does it give better and better approximation for the number $10/3$ note that to get exactly $10/3$ we have to keep on adding 3 with 3×10^{-1} with 3×10^{-2} and So on in fact we have to deal with infinitely many real numbers this cost of seeking better approximation in fact leads to the concept of sum of infinitely many real numbers however let us remark that there are some issues or trouble when we deal with some of infinitely many real numbers what are these issues in what follows i would like to illustrate these troubles with the help of some examples first of all note that while dealing with the question of finding sum of infinitely many real numbers we cannot keep on adding to see what comes out because there are infinite number of real numbers when we deal with finite number of real numbers say $a_1, a_2, a_3, \dots, a_n$ and when we want to find its sum first we can find $a_1 + a_2$ for instance to this sum we can add say a_3 and So on this process will terminate whereas while dealing with sum of infinitely many real numbers we cannot keep on adding and see what comes out that's one thing we should keep in mind with regard to question of finding some of infinitely many real numbers now let us deal with some specific example to see what are all the other issues suppose we want to find this infinite sum $1 + 2 + 4 + 8 + 16 + \dots$ i hope you can guess what's the pattern followed by the terms here in fact the n th summand here would be $1 \times 2^{n-1}$ in this sum the first number is 1 by 2 second number is 1 by 2 square third is 1 by 2 cube and

So on let us see whether we can find this infinite sum as i remarked you cannot first find the sum of 1 by 2 and 1 by 4 then add to this sum 1 by 8 and

So on because it is an infinite process but then let us view this problem slightly differently with the help of geometry you consider a unit square square whose side lengths are 1 we all know that its area is 1 square units let us half this square then area of the first half will be half square units let us again half the second half the area here will be one by four let us continue this halving process area of this figure will be 1 by 8 and

So on we can see that the areas of the smaller figures fill the area of unit square we started with thus geometrically we can observe that the sum of the areas namely $1 + 1/2 + 1/4 + 1/8 + 1/16 + \dots$ amounts to the total area of unit square we started with namely 1. thus we observe that $1 + 1/2 + 1/4 + 1/8 + 1/16 + \dots$ the sum of infinite number of real numbers is equal to 1. let us proceed with another example were we are interested to find this infinite sum $1 + 2 + 3 + 4 + \dots$ it's very clear that as and when we add more and more term the result increases that is $1 + 2 = 3$ when we add the 3 with this sum we will get 6 when we add 4 with the previous sum namely 6 we get 10 and

So on as we add more and more terms the sum increases we can see that by taking more and more terms the sum can be made larger than any pre-chosen value

So intuitively we observe that the sum $1 + 2 + 3 + 4 + \dots$ cannot be a finite value this is in contrast to the previous example were sum of infinitely many real values yielded a finite number namely 1. does when dealing with some of infinite number of real numbers we cannot always assert that ultimately this sum comes out to be a finite value in some cases it may in some cases it may not even to judge whether some of infinitely many real values will be finite or infinite is difficult for instance if we are concerned with this sum $1 + 1/2 + 1/3 + 1/4 + \dots$ it's not very clear whether ultimately this infinite sum amounts to a finite value or not

So what we observe is the following in contrast to sum of finitely many real values when dealing with sum of infinitely many real values whether we can come up with a finite value or not is not very clear it's one thing or one difference between sum of finite number of real numbers and sum of infinite number of real numbers we should not jump to the conclusion that sum of infinite number of real numbers will be always infinite no we can sometime have finite value also

So a natural question is how do we assign a definite meaning to an infinite sum how do we see whether an infinite sum comes out to be a finite value or we cannot find it these are the questions we have let me proceed with some more example to see other troubles that we may come across when dealing with sum of infinite number of real numbers let us consider this example as you see here we are dealing with sum of infinite number of real numbers which is alternatively positive and negative $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ and So on assume for a moment that this infinite sum comes out to be a finite value with that assumption let us proceed to find what is that value suppose we try to find this infinite sum with a grouping as follows with this particular grouping one can observe that each sum in the paranthesis is positive

So intuitively we feel that the given infinite sum amounts to a positive value on the other hand let us deal with same sum in a slightly different manner recall the infinite sum we are dealing with is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ etc suppose we rearrange the terms of given series as follows $-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \frac{1}{17} - \frac{1}{18} + \dots$ and

So on note that here before each positive term of the given series we supply a block of sufficiently many negative terms of the given series now let us find the sum of the numbers in these parentheses one at a time then it should be clear that the sum of given series can never be positive

So what we observe in this example is ordering of terms appearing in an infinite sum matters a lot with respect to a particular ordering the infinite sum may amount to one real number and with respect to another ordering the same infinite sum may amount to a different real number this is in contrast to finite case were sum of finitely many real values will be always same irrespective of the order in which we added it to sum up in contrast to sum of finitely many real values when dealing with sum of infinitely many real values the question of whether ultimately it represent a finite real number or an infinite real number is not very clear secondly the order in which we add infinite number of real numbers matters a lot thus with this example one should understand that when dealing with infinite number of real numbers and its summation one should always first order the real numbers appearing in that summation in other words order is important well a particular ordering corresponds to a sequence thus while dealing with sum of infinite number of real numbers one should not just start with a set of infinite number of real numbers but one should actually start with a sequence of real numbers once you start with a sequence a particular ordering is a priori given keeping all these examples and remarks in mind we are going to make some formal definitions to start with let us fix some notation suppose that we are supplied with finite number of real numbers let us say a_1, a_2, \dots, a_n and we would like to deal with the sum $a_1 + a_2 + \dots + a_n$ this finite sum which can be always found by normal addition and which always amounts to a finite value can be represented in a compact fashion using sigma notation that is by using uppercase greek letter sigma we denote $a_1 + a_2 + \dots + a_n$ in more compact fashion as follows $\sum_{i=1}^n a_i$ instead of writing the expanded form $a_1 + a_2 + \dots + a_n$ we can represent it using summation or $\sum_{i=1}^n a_i$ this variable i is called index of summation note that the index of summation is dummy in the following sense the sum $a_1 + a_2 + \dots + a_n$ can be written in compact fashion using the sigma notation $\sum_{i=1}^n a_i$ or using the sigma notation as $\sum_{j=1}^n a_j$ or as $\sum_{r=1}^n a_r$ the value of the index which indicates the first term in that summation here $i=1$ that is called lower limit and value of the index which represents the final term in that sum is called upper limit in this sigma notation one is the lower limit and n is the upper limit slightly more generally if we are supplied with a sequence $a_r, r=1$ to infinity in a more explicit form the sequence $a_1, a_2, a_3, a_4, \dots$

So on is given to you suppose that we want to find sum of the terms a_m, a_{m+1}, \dots, a_n we want to sum these terms that is we want to find $a_m + a_{m+1} + \dots + a_n$ in fact we are dealing with finite number of real numbers

So by normal addition we can find this this sum can be represented in a compact fashion as summation a_i i equal to m to n the lower limit should correspond to the first term in that sum and the upper limit should correspond to the last term in that sum this is just to fix a notation which will facilitate to save time and space let us illustrate with example this notation find summation j^2 j equal to 1 to 5 recall that this is a summation expressed in a compact fashion using greek letter sigma and what this stands for is 1 square plus 2 square plus 3 square plus 4 square plus 5 square do you see it 1 is the lower limit 5 is the upper limit index is j and you are finding sum of j^2 j is from 1 to 5 and this is 1 plus 4 plus 9 plus 16 plus 25 which is 55 again to fix this notation in your mind let us proceed with one more example find summation $(-1)^r$ r equal to 1 to 8 remember this notation is used for a sum of finite number of real numbers in this example index is r lower limit is 1 upper limit is 8 and summation is used to denote $(-1)^1$ that's the first term which is -1 plus $(-1)^2$ which is 1 plus $(-1)^3$ which is -1 plus $(-1)^4$ which is 1 and

So on upper limit is eight lower limit is one

So you are dealing with eight terms the last term is $(-1)^8$ which is one since we are dealing with finite number of real numbers this sum can be found in any order whichever is convenient for us

So let us group in this manner finally we get 0 now let us proceed with a definition this definition is motivated by the examples that we have seen already and the remarks that we made one thing is that while dealing with sum of infinitely many real numbers there should be an order in which we add the real numbers that is we should start with a sequence rather than just an infinite set secondly we cannot keep on adding and see what comes out when we are dealing with an infinite sum keeping all these in mind let us make some definitions given a sequence a_n the expression a_1 plus a_2 plus a_3 plus etc is called a series associated with the sequence a_n here is the definition for a series recall we told sequence and series are interchangeably used in day-to-day life both of them mean successive events or objects whereas in mathematics sequence is used for ordered list we did not see what a series stand for here is the definition given a sequence a_n you consider the expression a_1 plus a_2 plus a_3 plus etc this expression is what we mean by a series thus informally the distinction between sequence and series is that sequence is ordered list of numbers whereas series is a sum lot of questions remains here since we may deal with an infinite sum it's not clear whether this expression has a meaning or not in the sense whether this sum will ultimately give a finite value or not those questions can be answered later but for the moment i would like you to understand the definition of a series given a sequence the sum of its terms is what we refer to as a series we will proceed with concept of series in next few classes thank you

So much you