

so today i am going to introduce the concept of random variables uh

So let us recapitulate what we have done till now we have considered that there is a random experiment the set of all possible outcomes is called a sample space then any subset of the sample space is an event we have studied various methods of calculating probabilities of events we have given an axiomatic framework in that asymmetric framework we can calculate probabilities of various combinations of events for example there is a formula for union of the events what is the probability of conditional probability we have defined bayes theorem etc we have also studied the concept of independent events now what happens that many times we are not interested in the full description of the event rather we are interested in some numerical characteristic associated with that let's consider say a match say for example it's a badminton match now in a badminton match we can look at the outcome

So it is in the form of some score

So for example 21 19 that means the winning player in a game made 21 points and the losing player made 19 points if we consider a game of tennis then we can look at the points

So there are sets and the sets scores are given like 6 4 6 4 6 3 kind of thing r75

So actually the full duration of the match may have lot of facets that means how many ac's were served how many double faults were there but ultimately we look at the score similarly if you look at a cricket match then we may look at the scores scored by say certain players or the wickets taken by certain players that means to various events we are associating numbers ah this you can consider in other ways also for example let us look at some real life situation a patient goes to the doctor now he will get some medicines now after that taking the medicine the outcome is to be seen as whether he is getting cured or he is not getting cured similarly from the doctor's point of view we can see that he is treating patients

So for example in a day he treats 20 patients now out of those 20 patients how many actually got benefited say maybe 18 got benefited two did not get benefited

So if we look at this kind of thing

So we are actually looking at certain numbers associated with events now this association can be described by the concept of random variables let us look at simplest example suppose we consider a basketball game and we see how many successful basket hits are done by a player a

So in that case then this number say x is a numerical valued variable now since the playing of the match and how many he will be successfully hitting that itself is a random event therefore this becomes a random variable since the success in hitting the basket is random we call this as a random variable

So let us consider say we consider tosses of three coins okay now the outcomes are the sample space can be written as all the three heads two heads and one tail which can be in the form of hht hth and thh or you can have two tails or you can have all the three tails but we are not interested in the actual outcome rather we are interested in how many heads are observed or how many tails are observed

So i define from this experiment let x denote the number of tails then x can take values 0 1 2 or 3. in fact now you can see i want to put a value corresponding to each element of the sample space

So for example i can say x of h h is 0 because there is no tail here x of h h t is 1 because there is one tail here similarly if i look at x of h t h then this is equal to 1 if i put x of t h h then this is 1 then if i put x of h t t then there are two tails what is x of t h t that is 2 x of t t h is equal to 2 and x of t t t is equal to 3

So what we have done to each element of the sample space we have associated a real number So this assignment by using a function we are using the function x here this x is called a random variable thus x assigns a real number to each element of the sample space

So x is called a random variable

So let me give a formal definition of the random variable a random variable x is a real valued function defined on the sample space let us follow the popular terminology which you use in mathematics how do you write a function you write a function like f is from a to b kind of thing

So that means here x is from s to the set of real numbers s is a sample space and r is the set of all real numbers

So x is actually a function ah but in probabilistic terminology the convention is to call it a random variable ah okay

So the name can also be explained it's a variable because depending upon different outcomes it takes different values and those outcomes are coming from a random experiment therefore it is called a random variable mathematically it is a function that means to each ω belonging to S $X(\omega)$ is a real number

So examples of random variables can be for example if i look at height of an adult male So this will be some number

So taking the extreme case that somebody can be say four feet

So that means say around 120 centimeter and somebody may be as high as say eight meter taking the extreme uh case uh eight feet

So maybe is ah 240 centimeter kind of thing

So we can have the set of values x taking values between say 120 to 240 centimeter suppose i consider age of a person

So age of a person can be from 0 to any

So suppose i am considering in numbers

So maybe the oldest known person may be maximum 120 years of age

So you can put 0 to 125 years number of attempts needed to hit a target what are the possible values of this it can take value one you can hit in one you cannot hit in two and

So on you can see then the type i gave an example where the random variable is taking values 0 1 2 r 3 i have given an example here where the values taken are in an interval that means uncountably infinite number of values here you can see number of attempts So 1 2 3 and

So on it is countably infinite number of values

So a random variable based on this description can be described as either the one which takes a finite or countably infinite number of values that means you can write 1 2 3 and

So on say up to n r 1 2 3 and

So on infinitely many that means you can say countably infinite number of values or if you say height age weight life time etc then these are all examples of continuous random variables because the random variable takes values over an interval in your class 11 and 12 you have in the syllabus discrete random variables

So i will describe in detail uh the probability distribution of a discrete random variable how to find out its expectation or the mean etc

So if a random variable takes a finite or countably infinite number of values then it is called a discrete random variable if the random variable X takes values over an interval it is called a continuous random variable actually there are finer distinctions based on the representation of the probability distribution ah but at this stage we will take these definitions as the definition of discrete and continuous random variables when you will go to the advanced classes then you will learn the more rigorous definitions of the random variable for example it is a measurable function

So but in the class eleventh and twelfth level uh we do not study into that depth uh

So you should understand the method of uh finding out the probability distribution for a discrete random variable that is in your syllabus

So we will spend some time on that

So thus we can say that for a discrete random variable X the set of all possible values can be described by

So i'll use some notation say E is equal to say $X = \{x_1, x_2, \dots\}$ and

So on x_n are x_1, x_2, \dots and

So on

So either you have a finite number of elements say n elements or you have an infinite number of elements but they can be counted

So it is countably infinite number now in the original random experiment uh based on the description of the experiment various events or various outcomes are allocated certain probabilities now when those events are transformed to values

So each element is transformed to a value using the random variable then the corresponding probabilities can be associated with those values that gives rise to a discrete probability distribution

So let me talk about this the probability distribution of a discrete random variable is an assignment of probabilities to each value that the random variable X may take

So let me keep this sheet here to consider what are the possibilities

So if i consider the random variable X taking values x_1, x_2, \dots, x_n then if i allocate the probability p of x_1

So we can write probability x is equal to x_1 is equal to say p_1 probability of x is equal to x_2 is equal to p_2 and

So on probability of x is equal to x_n is equal to p_n now you can see that all the possibilities of the random sample are allocated the values x_1, x_2, \dots, x_n now in the original sample space when the allocation of the probabilities was there there were some conditions which were satisfied for example sum of all the probabilities is 1. now those probabilities have been transformed to p_1, p_2, \dots, p_n therefore the sum of p_i 's must be equal to 1 also these are all probabilities

So all the probabilities must be non negative now in the discrete random variable we will talk about exactly those values which are assigned

So that means the probabilities are actually positives

So we can say that p_i is positive for all i and $\sum_{i=1}^n p_i$ is equal to 1 then this p_1, p_2, \dots, p_n is called the probability distribution of random variable x

So here we are dealing with the discrete random variable x and i am assigning the probabilities p_1 to x_1, p_2 to x_2, \dots, p_n then this p_1, p_2, \dots, p_n is called the probability distribution of the random variable x let me calculate it for one case consider this example of tossing of three coins if i assume the coins to be fair then we can calculate the probabilities here of each value consider the experiment of tossing of three fair coins and here x is the number of tails then let us calculate the probability distribution what is the probability that x is equal to zero now x is equal to zero is corresponding to when all the three heads are observed

So it is actually equal to the probability of the possibility $h h h$

So this is 1 by 8 . similarly if i look at what is the probability of x is equal to one then from the experiment you can see that x is equal to 1 is when one tail is observed

So $h h t, h t h$ and $t h h$

So this we can write as $h h t, h t h$ and $t h h$

So this probability is 3 by 8 .

So we have calculated the probability of x is equal to 1 similarly if i look at what is the probability of x is equal to 2 then you can see that x takes value 2 when two tails are observed that is corresponding to $h t t, t h t$ and $t t h$

So that is probability of $h t t, t h t$ and $t t h$ again you can see that this probability is equal to 3 by 8 similarly you can see p_3 that is the probability x is equal to 3

So x is equal to 3 when all the 3 tails are there

So it is the probability of $t t t$ that is 1 by 8 .

So now you can see i have calculated the probabilities corresponding to all the possible values of the random variable for x is equal to zero p_0 is one by eight p_1 that is the probability x equal to one is three by eight p_2 that is probability x equal to two is three by 8 and p_3 that is the probability x equal to 3 is 1 by 8 .

So if you look at the sum of these it is equal to 1 by 1 plus 3 plus 3 plus 1 that is 8 by 8 .

So that is 1 you are having p_0 plus p_1 plus p_2 plus p_3 is equal to 1

So this is a valid probability distribution here let me take another example suppose there are 10 bulbs in a pack out of which three are defective a customer buys two of these at random okay in a pack of bulbs there are ten bulbs out of which three are defective and a customer buys two of these at random now certainly when he is buying two there can be some defectives there

So let x be the number of defectives purchased by the customer then what are the possible values of x x can take values

So out of two maybe all are good

So that means zero defectives one may be defective or both may be defective

So this is x is a discrete random variable now we want to calculate the probability distribution of x that means what is the probability x equal to 0 what is the probability x equal to 1 and what is the probability x is equal to 2

So in order to calculate this let us look at the calculation of various possibilities if we are considering choosing of two bulbs from a pack of ten bulbs then the total number of possibilities is ${}^{10}C_2$ now if i say that none of them is defective that means customer chose 2 and he got both good ones now out of this 10 bulbs 7 are good that means his selection is from those 7

So it is 7C_2 divided by ${}^{10}C_2$ that means the favorable number of cases is 7C_2 and the total number of cases is ${}^{10}C_2$ now this can be easily simplified

So this is giving 21 by 45 ah this can be further simplified i have not simplified

intentionally just to show that the sum is all right similarly let us consider p_1 what is the probability that x is equal to one once again the total number of possibilities is ten C_2 now if one is defective that means one is non-defective So that means out of seven he gets one good one and from the three defectives he chooses one

So the favorable number of cases is seven C_1 into $3 C_1$ divided by $10 C_2$

So that is again 21 by 45

So that you can actually write as 7 by 15 this is also 7 by 15 and p_2 that is the probability that x equal to 2 that means here got both the defective ones

So $3 C_2$ divided by $10 C_2$ that is equal to 3 by 45 that is equal to 1 by 15 this is the probability distribution of the number of defectives you can see here 7 by 15 plus 7 by 15 plus 1 by 15 the sum is equal to 1

So this is a valid probability distribution of the discrete random variable x which is defined as the number of defectives in the purchase by the customer let me take one more example here a card is drawn at random from a well shuffled deck of 52 cards that means the complete set is there of 52 cards from there one card is drawn at random if the card drawn is any number between 2 to 10 then its score is that number that means if we draw a 2 then the score is allocated as 2 . if we draw a 5 then the score allocated is 5 . if the card drawn is king queen or jack its a score is 15 . if an ace is drawn its score is 18 okay

So let us look at the random variable let x denote the score then what are the possible values of x the possible values of x are 2 3 up to 10 if the king queen or the jack is drawn the score allocated is 15 and if an ace is drawn the score allocated is 18 .

So the values that x can take is 2 3 up to 10 15 and 18 .

So it's a discrete random variable let us calculate the probability distribution of this So what is p_2 that is probability that x is equal to 2 there are 4 cards which carry the value two

So four by fifty two that is equal to one by thirteen similarly if i look at p_3 again there are four cards that carry the value 3

So it is 4 by 52 that is equal to 1 by 13 . up to p_{10} you will have same value if i consider p_{15} that is the probability x equal to 15 now 15 is recorded then 3 cards are there king queen and jack there are 12 such cards

So you get 12 divided by 52 that is equal to 3 by 13 and the probability of 18 that is probability x equal to 1 ah x is equal to 18 that is when an s is observed

So again it is 1 by 13 .

So this is the probability distribution of this random variable x you can look at the sum you have 9 values here 9 by 13 plus 3 by 13 plus 1 by 13 that is equal to 1 . now if a random variables probability distribution is there we can calculate various probabilities So for example if i am looking at the number of tails i can ask what is the probability that an odd number of tails are observed

So for example odd number of tails will be probability x equal to 1 plus probability x equal to 3 that means 3 by 8 plus 1 by 8 i can ask what is the probability that x is less than or equal to 2

So if i say probability x less than or equal to 2 it is probability x equal to 0 plus probability x equal to one plus probability x equal to two that is one by eight plus three by eight plus three by eight that is seven by eight the point i am trying to make is that given a probability distribution we can resolve the probability statements corresponding to that random variable

So let me just calculate few of these probabilities here we want the probability that the score is at least 10 that means what is the probability x is greater than or equal to 10 that is equal to probability x is equal to 10 plus probability x equal to fifteen plus probability x equal to eighteen these are the possible values that the random variable can take which are not less than ten

So this is equal to 1 by 13 plus 3 by 13 plus 1 by 13 that is 5 by 13 .

apart from calculating the probabilities of various values of the random variable one can also calculate its mean or you can say uh the central point of the distribution ah if you remember your statistics portion given values x_1 x_2 x_n in the sample you are calculating arithmetic mean which you are calculating by looking at x_1 plus x_2 plus x_n by n or if the frequency distribution is given for x_1 you have frequency f_1 for x_2 you have frequency f_2 for x_n you have frequency f_n then you are calculating $x_1 f_1$ plus $x_2 f_2$ plus $x_n f_n$ divided by $\sum f_i$ and what was the purpose of that it gives you a measure of

central tendency for that data similarly when the random variable is taking values x_1, x_2, \dots, x_n with probabilities p_1, p_2, \dots, p_n we can calculate x_1 into p_1 plus x_2 into p_2 plus x_n into p_n this is called the mean of the discrete distribution of the random variable x let us define that let x be a discrete random variable with possible values in E that is x_1, x_2, \dots, x_n and the corresponding probability distribution p_1, p_2, \dots, p_n the mean of x are expectation of x our expected value of x is defined by

So this is the notation expectation value is refined as $E(x)$ that is x_1 into p_1 plus x_2 into p_2 plus and

So on x_n into p_n that is equal to $\sum_{i=1}^n x_i p_i$ i is equal to 1 to n

So you can see here what are the values that the random variable takes we consider those values and multiply them by the corresponding probabilities and sum it

So then it becomes the mean or the expectation of the random variable

So let us look at the distributions that we have done

So we had this distribution of the number of tails in the three tosses of a coin let us look at what is the mean here distribution of number of tails

So for your reference i will i again write here it was p_0 was 1 by 8 p_1 was 3 by 8 p_2 was 3 by 8 and p_3 was 1 by 8 .

So the mean or the expected value is 0 into 1 by 8 plus 1 into 3 by 8 plus 2 into 3 by 8 plus 3 into 1 by 8 that is equal to twelve by eight that is equal to three by two ah now one may wonder that what is the meaning of uh this i am getting a fraction actually the number of tails is $0, 1, 2, 3$

So that is an integer value the expected value or the mean does not mean that it has to be one of those values but it is some intermediate value here the values are $0, 1, 2$

actually you can see here if i consider the plotting here

So suppose i put a 0 here 1 here 2 here and 3 here then on this side we are putting p_i So 1 by 8 and this is three by eight then this is three by eight and this is one by eight So this is one by eight this is three by eight this is three by eight and this is again one by eight okay

So you can see here that the value that i am getting as the mean is 3 by 2 that is coming here that is something like a mid value and that is happening because it is a distribution which is symmetric that means from both the ends i am giving equal probability to 0 and 3 and equal probability to 1 and 2 .

So that is why the mean value is turning out to be in the middle let us look at the other example of the number of defective bulbs number of defective bulbs

So here p_0 was 7 by 15 p_1 was 7 by 15 and p_2 is equal to 1 by 15 .

So expectation of x becomes zero into seven by fifteen plus one into seven by fifteen plus two into one by fifteen that is equal to 9 by 15 that is equal to 3 by 5 again you can see the number of defectives is $0, 1$ or 2 but the mean value or the average value is not an integer actually it is a fraction which is below this if you plot the distribution here at 0 you are having 7 by 15 at 1 you are having seven by fifteen and at two you are having one by fifteen

So this average is coming somewhere here that is three by five somewhere here you are getting let us look at another example of the score of the card drawn

So here we have this probability distribution let us look at the mean score of the card

So here the distribution was let me rewrite the distribution probability x is equal to i was 1 by 13 for i is equal to 2 up to 10 probability x is equal to 15 was 3 by 13 and probability x is equal to 18 was 1 by 13 .

So expectation of x becomes i into 1 by 13 for i is equal to 2 to 10 plus 15 into 3 by 13 plus 18 into 1 by 13 ah this sum you can do by the summation of the positive integers formula like the sum from 1 to 10 you know that it is n into n plus 1 by 2

So that is 10 into 11 by 2 that is 55 and the first term is not there

So it will become 54 .

So 54 plus 45 plus 18 by 13 .

So that is equal to nine expectation x is equal to nine that is the expected score of the card is nine now let me also introduce another quantity that is called the variance

So i have explained that expectation is something like average value or the mean value but we may also like to see whether the values are distributed more that means they have lot of variation or they have less variation let us consider the concept of variability

So let us take probability x is equal to minus 1 is equal to probability x equal to 0 is equal to probability x is equal to 1 it is equal to say 1 by 3 okay that means minus $1, 0$ and 1 each of them is having equal probability 1 by 3 okay

So if you look at the expectation of x that is equal to $\frac{1}{3}(-1) + \frac{0}{3} + \frac{1}{3}(1) = 0$. ah let us take another example probability x is equal to -2 is equal to $\frac{1}{3}$ probability x equal to 0 is equal to $\frac{0}{3}$ probability x equal to 2 is equal to $\frac{1}{3}$ again you can see expectation of x is equal to $\frac{1}{3}(-2) + \frac{0}{3} + \frac{1}{3}(2) = 0$ but if you plot it how you looks like -2 and 2 one by three one by three one by three So if i compare these two graphs you can see that the random variable x here takes values which are having more variation compared to this random variable let me rename these variables

So i call this as x_1 random variable and this random variable i call x_2 random variable then what we are observing is that expectation of x_1 and expectation of x_2 is 0 and the random variable x_1 gives equal probability to -1 , 0 and 1 the random variable x_2 gives equal probabilities to values -2 , 0 and 2 . now visibly i can see that this values are near these values are slightly away there is a more variation here in order to measure that we introduce a concept called variance

So let me define that variance of a random variable x is defined by $E[(x - \mu)^2]$ So we call it where μ of x or sometimes it is written as V of x okay it is nothing but expectation

So expectation terminology i have already introduced

So let us see expectation of what

So $E[x - \mu]^2$

So let me define where μ is used as a notation for expectation of x

So now you can see what i am trying to look at if μ is the mean of x then i am looking at how much variation is of the random variable from the mean value this is what was little bit of intriguing in these two examples if you look at here the mean is 0 and other values are 1 and -1 here the mean is 0 and the other values are 2 and -2 . So obviously these values are farther away from the mean value compared to this

So let us look at this variance of x_1 and variance of x_2

So variance of x_1 that is $E[x_1 - \mu_1]^2$ obviously here μ_1 is 0

So this value is equal to $E[x_1 - 0]^2$ that is expectation of x_1 square i already introduced the formula for expectation it is value multiplied by the probability

So what are the values of $x_1 - 0$

So -1 square becomes 1 the probability is $\frac{1}{3}$ 0 one by three plus 1

So 1 square is 1 one by three

So this value is equal to $\frac{2}{3}$ that is the variance of x_1 let us look at the same thing for the random variable x_2

So for the random variable x_2 variance of x_2 that is $E[x_2 - \mu_2]^2$ where μ_2 is nothing but the expectation of x_2 once again expectation of x_2 is 0

So this becomes expectation of x_2 square x_2 is taking values -2

So -2 square is 4 into the probability of x_2 is equal to $\frac{1}{3}$ plus 0 into the probability plus 2 square is 4 into $\frac{1}{3}$ it becomes $\frac{8}{3}$

So let us compare the variance of x_1 and the variance of x_2

So you can see here variance of x_1 is $\frac{2}{3}$ and variance of x_2 is $\frac{8}{3}$ naturally x_2 is having more variation than x_1

So this concept of looking at that there is some variation in the random variables

So whether there is a less variation or there is a more variation this concept can be formally studied using the term variance

So thus variance of x_2 is greater than variance of x_1 thus we can see that random variable x_2 has more variation than random variable x_1

So we can actually calculate variance in various examples that we have done

So number of tails

So we had here p_0 was $\frac{1}{8}$ p_1 was $\frac{3}{8}$ p_2 was $\frac{3}{8}$ and p_3 was $\frac{1}{8}$ and expectation x which we call μ that was equal to $\frac{3}{2}$

So let us calculate here variance

So variance of x is equal to $E[x - \mu]^2$ that is expectation of x minus $\frac{3}{2}$ square we have to substitute the values of x

So that is 0 minus 3 by 2 square into the probability that is 1 by 8 plus 1 minus 3 by 2 square into 3 by 8 plus 2 minus 3 by 2 square into 3 by 8 plus 1 by 8 sorry 3 minus 3 by 2 square into 1 by 8 .

So this you can easily calculate it is 9 by 4 into 1 by 8 plus 1 by 4 into 3 by 8 plus 1 by 4 into 3 by 8 plus 9 by 4 into 1 by 8 .

So these values turn out to be 24 by 32 that is equal to 3 by 4 .

So i have given the example that how the variability of the distribution can be calculated we have introduced the discrete random variables the random variables which take a finite or a countably infinite number of values although all the examples that i have taken there we have taken finite number of values i will discuss in detail some other examples where even infinite number of values are permitted and i have introduced the concept of expectation or the mean and the variance in the next class i will further enhance this concept by using the spatial discrete random variable called a binomial you