in the last lecture i have given several problems of probability the purpose of these problems was to show the applications of various rules for example addition rule the concept of conditional probability the theorem of total probability and the bayes theorem today also i will spend some more time on solving problems many of these problems are also taken from question papers of some competitive examinations like joint entrance examination and some other examinations again you can see that most of these problems will utilize all the concepts that we have discussed till now also i would advise that please go through the chapter on permutations and combinations because some of the probability problems involve the concepts of permutations and combinations

So it would be better if the students are well prepared for that

So let me start with some problems the probabilities of observing a six when three dice

So let me name them d1 d2 d3 are tossed independently are 1 by 6 1 by 5 1 by 4 respectively ah

So that means if the die d1 is tossed then the probability of observing a six is one by six if the die d2 is tossed then the probability of observing a six is one by five that means it is a biased die similarly if d3 is tossed then the probability of observing s six is one by four again it is a biased die and that dice are tossed independently now the problem is like this a die

So that means out of d1 d2 d3 is chosen randomly

So that the probability of choosing d i is proportional to 1 by i for i is equal to 1 2 3 this die is tossed what is the probability that a a6 is observed if a6 is observed what is the probability that di d2 was chosen

So let me again repeat the problem there are three dice if dice d1 is tossed then the probability of head is 1 by 6 if the die d2 is tossed then the probability of 6 not 1 by not the head it is 1 by 5 if the die d3 is tossed then the probability of six is one by four now one die is chosen at random but the probability of each die is not one by three the probability is actually proportional to one by i that means if d i is chosen the probability will be some alpha by i for i is equal to 1 to 3 then this particular die which has been chosen is tasked and then some question is asked that is what is the probability that is 6 is observed if a 6 is observed what is the probability that the die d2 was chosen

So let us look at the solution here let e denote the event that a 6 is observed ok a 6 is observed and let a i be the event that di d i is chosen for i is equal to 1 to 3

So then if we consider probability of a i

So probability of a i is some alpha by i for i is equal to 1 2 3 now we are choosing among from these three dice only therefore the sum of probability of a one plus probability of a two plus probability of a three it will be equal to one that means these events a 1 a 2 a 3 are exhaustive

So you will have probability of a 1 plus probability of a 2 plus probability of a 3 is equal to 1

So probability of a 1 is alpha by 1 probability of a 2 is alpha by 2 and the probability of a 3 is alpha by 3

So that is equal to 1

So if you simplify this this is becoming 11 by 6

So you get alpha is equal to 6 by 11

So if we substitute back in the probability of a i i get probability of a i is equal to 6 by 11 sorry probability of a 1 probability of a 2 is 3 by 11 and the probability of a 3 is equal to 2 by 11.

So now what is the probability of e given a one that is one by six the probability of observing is six when second coin is chosen is one by five and the probability of observing a six when the third dice is chosen is one by four now by theorem of total probability probability of e is equal to sigma probability of e given a i into probability of a i i is equal to 1 to 3.

So all the values are available to us

So it is becoming probability of e given a 1 that is 1 by 6 into probability of a 1 that is 6 by 11 probability of e given a 2 that is 1 by 5 into probability of a two that is three by eleven plus probability of e given a three that is one by four into probability of a three that is two by eleven

So if we add up these terms we get 21 by 110

So the probability of observing a 6 is 21 by 110 obviously since the at least two of the dice were not unbiased they were not fair dice therefore you can see the probability of a

six is not one by six it is 21 by 110

So now let us look at the second part of it if a 6 is observed what is the probability that the die d2 was chosen that means we want what is the probability of a 2 given that a 6 is observed

So this is a direct application of bayes theorem

So this is equal to probability of e given a 2 into probability of a 2 divided by probability of e

So that is equal to 1 by 5 into 3 by 11 divided by twenty one by one one zero

So this can be easily simplified and you get it as equal to two by seven

So this is an application of by bayes theorem

So let us take another problem of a similar nature

So there are three shooters let us call them s1 s2 and s3

So they hit a target successfully with respective probabilities two by five one by three and three by seven

So the probability that the shooter s one will hit the target is two by five the probability that the shooter s 2 will hit the target is 1 by 3 the probability that the shooter s 3 will hit the target is 3 by 7.

So they shoot simultaneously and independently of each other what is the conditional probability that s2 missed the target given that there were exactly two hits

So let us define the events we define event bi that s i hits the target for i is equal to 1 2 3 then the events b1 b2 b3 are independent because it is given that the shooters suit independently of each other b1 b2 b3 are independent also we are given what is the probability of b one what is the probability of b two what is the probability of b three that is two by five one by three and three by seven we can write it here then probability of b 1 is equal to 2 by 5 probability of b 2 is 1 by 3 and probability of b 3 is equal to 3 by 7

So let us consider a to be the event that there are two hits then what is the event a then the event a is okay we have to express it in terms of b1 b2 and b3

So if there are two hits that mean we can have the situation that b1 and b2 hit and b3 does not hit similarly we can have the first and third hit second does not hit r second and third hit and the first does not hit

So we can say a is b1 ah intersection b2 intersection b3 complement

So it means the shooter 1 and 2 they hit and the third one misses

So i have written b three complement union we can have b1 and b2 complement and b3

So it means the first and the third shooter are successful whereas the second shooter is unsuccessful union b1 complement intersection b2 intersection b3 it means that the second shooter and the third shooter they are successful whereas the first shooter is not successful

So what we have done we have represented the event a as a union of

So this is one event this is one event and this is one event these three are disjoint events why they are disjoint because in this one for example b3 complement is there and here b3 is there

So certainly this one is a subset of b3 this one is a subset of b3 complement because these are intersections of the events

So these two cannot have any element in common similarly if i take second and the third one here b1 complement is there here here b1 is there and here b1 complement is there

So again they are disjoined similarly if i look at the first and the third

So here b3 complement is there here b3 is there

So they will be independent

So if i consider probability of a then i will get it is equal to probability of b 1 intersection b 2 intersection b3 complement plus probability of b1 intersection b2 complement intersection b3 plus probability of b1 complement intersection b2 intersection b3 again ah in one of the earlier problems i have shown that if two events are independent then their complementations are one complement or the other they are all independent

So if b1 b2 b3 are independent i will have b1 b2 and b3 complement independent b1 b2 complement and b3 independent and b1 complement b2 and b3 independent

So the probabilities of these intersections become product of the probabilities

So for example the first one becomes probability of b 1 into probability of b 2 into probability of b three complement similarly if i look at the next one then it is probability of b one into probability of b two complement into probability of b three

plus probability of b one complement into probability of b two into probability of b three ah now all the values of this probabilities of b one b two and b three are available
So we can substitute them here probability of b three is three by seven
So probability of b three complement will become four by seven
So this term can be easily simplified and we get it to be equal to 29 by 1 0 five
So the probability that there are exactly two hits is equal to twenty nine by one zero five now if i look at the question that is asked here what is the conditional probability that s 2 missed the target given that there were exactly 2 hits
So if i say s 2 makes the target
So b 2 is the event that s 2 hits the target
So b 2 complement
So we have to then calculate the probability of the required probability is equal to probability of b 2 complement given a that is equal to probability of b two complement intersection a divided by probability of a now what is b two complement intersection a
So if i look at the event a
So if i put b2 complement intersection a then it will become b2 complement intersection with the first term union b2 complement intersection with the second one union b2 complement intersection with the third one now you can notice here that in the first term it is b2
So if i take intersection with b2 complement i will get phi in the third one also it is b2
So if i take intersection with b2 complement i will get phi
So i will get only the term b1 intersection b2 complement intersection b3 here
So this event is equal to probability of b2 complement intersection b1 intersection b3 divided by probability of a
So the term that we have already simplified divided by probability of a
So this term is equal to 2 by 5 into 2 by 3 into 3 by 7 divided by 29 by 1 0 5
So this value turns out to be ah this is equal to 12 divided by 29 because this one zero five and one zero five will get cancelled out and here we get 12 by 29
So the conditional probability of the event that the shooter s 2 miss the target given that the the exactly two shooters were able to uh shoot that is equal to 12 by 29
So here this is an example of conditional probability as well as here i have used the concept of union of disjoint events and the concept of independence of events
So one extra thing that has been used you please note here that if any set of events are independent then if i include complementations of some of the events then they are also independent as i have mentioned in the beginning b 1 b 2 b 3 are independent but here i am using independence of b1 b2 and b3 complement independence of b1 b2 complement and b3 and b1 complement b2 and b3
So you note here that although these things look extremely simple because we are able to use the set theoretic notation but there are in each problem several concepts that we are being applied simultaneously ah let me give another example of the conditional probability
So a computer producing factory has only two plants say p1 and t2 plant t1 produces 20 percent and plant t2 produces 80 percent of the total computers produced ah seven percent of the computers produced by the factory turn out to be defective further it is known that probability that computer is defective given that it is produced in plant p1 is 10 times the probability that the computer is defective given that it is produced in plant t2 now a computer produced in the factory is randomly selected and it is not defective what is the probability that it is produced in plant t2
So the language is long
So let me read out the problem once again
So there is a factory which produces computers and the factory has two plants
So they are named as t1 t2 the total products 20 percent is produced by plant t1 and remaining 80 percent is produced by plant t2 also it is found that 7 percent of the product is defective and additional information that is also given that the computers produced by plant t one have 10 times the probability of defective as it compared to that it is produced in plant t2
So now if a computer is produced in the factory is randomly selected and it is found to be non then what is the probability that it is produced in t2
So let us define the events now let a be the event that the computer is defective and i

use the notation b_i that it is produced in plant t_i for i is equal to one two then it is given that probability of b_1 is one by five and probability of b_2 is equal to four by five because it is given that plant t_1 produces twenty percent

So that is one by five and plant t_2 produces eighty percent supply probability of b_2 is four by five

So it is also given that the probability of defective is seven by hundred

So if we apply the formula probability of a that is equal to probability of a given b_1 into probability of b_1 plus probability of a given b_2 into probability of b_2 that is by theorem of total probability you can note here that the condition for the theorem of total probability are satisfied here because you have two mutually exclusive and exhaustive events b_1 and b_2 and then there is an additional event a and the probabilities of a given b_1 a given b_2 etcetera can be used here now you are given that this condition probability that the computer is defective now this is a a given b_1 is 10 times probability of a given b_2

So let us make some notation here ah let us take say probability of a given b_2 is equal to say alpha then probability of a given b_1 it will become 10 alpha from this condition probability of a given b_1 is equal to ten times probability of a given b_2

So if i choose probability of a given b_2 to be alpha then probability of a given becomes n alpha

So now ah let me call it one

So in equation number one let us substitute this ah left hand side is probability of a that is 7 by 100 is equal to probability of a given b_1 that is 10 alpha into probability of b_1 that is one by five plus probability of a given b_2 that is alpha into probability of b_2 that is four by five

So that is equal to 14 alpha by 5

So we can easily simplify this that will give me alpha is equal to 1 by 40

So i have actually evaluated probability of a given b_2 as 1 by 40 and probability of a given b_1 as 1 by 4 probability of a given b_2 it becomes one by forty and probability of a given b_1 becomes one by four that is ten times this ok

So now what is the question that is asked what is the probability that it is produced in t_2 given that it is not defective that means in terms of events if i write the required probability is probability of b_2 it is produced in plant t_2 given that it is not defective that is b_2 given a compliment

So here i can apply again the conditional probability

So that is probability of b_2 intersection a compliment divided by probability of a complement and this is nothing but probability of a complement given b_2 into probability of b_2 divided by probability of a complement you can say it is bayes theorem or it is multiplication rule

So probability of a complement given b_2 that you can calculate from here because probability of a given b_2 is there

So this is becoming 1 minus 1 by 40 into probability of b_2 that is 4 by 5 divided by probability of a compliment

So probability of a is 7 by 100

So this is nothing but 93 by 100

So this term we can easily simplify and we get 78 by 93 ah

So one thing that you can observe here that the probability of b_1 is actually 1 by 5 and the probability of b_2 is 4 by 5 but if i am looking at the event that it is not defective then the probability of b_2 is 78 by 93

So that has become more than three by four the reason is that the number of defective items from the second plant is much less

So therefore this probability has become more in this problem you can notice i have used several concepts one is that concept of total probability then we have used the theorem of total probability to start with a certain equation which gives me the original conditional probabilities then i have applied the concept of conditional probability or you can say base theorem to evaluate the final probabilities ah let me now ah give one example of enumerative probability that means we have to count the number of cases

So suppose there are there are let me call them again d_1 and t_2 and they play two games against each other ok then probability that t_1 wins a game that is one by two probability the game is draw is one by six and the probability that t_2 wins the game wins a game is say one by three okay

So they are playing against each other

So this event even wins is also equivalent to saying t2 loses and if the game is drawn means both are not winning or you can say both are not losing similarly t2 wins a game means t1 actually loses the game because they are actually playing against each other uh the winning team gets three points the losing team zero points and for a draw both teams get one point each

So let us use some notation let x be the total points by team p1 and y be the total points by team t2 after two games then find say what is the probability of x is equal to y what is the probability of x greater than y what is the probability x less than y ah as i mentioned this is a case of enumerative probability we have to actually count the cases in which we get ah these events

So for example if i say x is equal to i since there are two games in what way x and y will be same that means the total points by team t1 and total points by t2 are same

So that is possible if both games are drawn or the first game or you can say one game is one by say t1 and another game is one by t2

So we can say t1 wins one game and t2 wins one game

So in this case also both will get three points in the first case if both the games are drawn both the teams get two points

So here x and y both are 2 and here x and y both are 3.

So these are the case when x is equal to y again you can see i have used the mutually exclusive and exhaustive events here the event that both the games are drawn or the event that t1 wins one game and t2 wins one game these two are mutually exclusive and exhaustive therefore the probability of the union is equal to the sum of the probabilities now in order to evaluate that both the games are drawn we have to actually look at the probability of one game being drawn and the second grain being drawn now this can be calculated only if i make the assumption that the games are played independently

So we have to write that assumption that the games are played independently

So in that case let us calculate this probability both the games are drawn

So the probability of a draw is one by six

So it becomes one by six and in the second game also the same outcome

So one by six

So here i have made use of the independence in the second case t1 wins one game

So the probability is half and t two wins one game the probability is one by three but here you can choose the order in which the games are played for example the first game may be one by t one and the second may be by t two or the reverse

So there are two such cases

So i have to multiply by two

So this we can easily evaluate it is equal to thirteen by thirty six similarly let us look at what is the probability of x greater than y

So x greater than y means t1 wins both games are t1 wins one game and one game is drawn now there is no other possibility because if t2 wins then his points will become equal to t1 or it will become more

So let us look at if t1 wins both the games his probability of winning is 1 by 2 into one by two he wins one game probability is half and drawing is one by six but again here you can choose the order in which t1 will win and the game is drawn

So there are two such possibilities

So if you add these two you get 5 by 12 you can say it is 15 by 36

So if you calculate what is the probability of x less than y then that is equal to 1 minus probability of x is equal to y and probability of x greater than y that is equal to 2 by 9 that is you can say 8 by 36

So we have calculated the probabilities of all options in this problem i have made use of the concept of independence the concept of mutually exclusive and exhaustive events let us look at another problem in which again i have to count the number of cases

So box 1

So this is again a problem which is from one of the joint entrance examination question papers if you see the old question papers this problem is there i will show you how the solution is described

So box one contains three cards bearing numbers one two three box 2 contains 5 cards bearing numbers one two three four five and box three contains seven cards bearing numbers one two three four five six and seven a card is drawn at random from each of the boxes let x i denote the number of the card drawn from the i th box for i is equal to 1 to 3 find the probability that x one plus x two plus x three is odd find the probability

that x1 x2 x3 are in an arithmetic progression which we use the term ap okay
So in box one there are three cards in box two there are five cards and in box three there are seven cards
So from each box when we are choosing the card then from box one there are three possible ways from the box two there are five possible ways and the from box three there are seven possible ways
So the total number of total number of ways of choosing or drawing the cards x 1 x 2 x 3 that is 3 into 5 into 7 that is equal to one hundred five
So now in part one we have to look at the number of cases in which x one plus x two plus x three is odd
So now see one thing is that we start writing down from one you can have one then second you have one then from the third one you have one
So like that you can start counting but it will take long time
So let us develop a systematic number theoretic approach
So we can say for example in x one you have only three possibilities
So i can have x one as x one can take 3 values
So if x 1 is equal to 1 or 3 and then you want x 2 plus x 3 that should be even because if x one is odd then if you add an even number then you will get an odd number
So then x two plus x three must be even similarly if x one is equal to two then x 2 plus x 3 must be odd
So let us look at this is much easy to look at
So either x 2 and x 3 are even
So that is possible in how many ways for x two you have even possibilities two and four and for x three you have even possibilities two four and six
So there are total two into three that is six cases or you can have x two and x three both are odd because sum of two odds is even
So now how many possibilities for x two r you have one three five that is three possibilities and for x three one three five and seven that is four possibilities
So that are total three into four that is twelve cases
So total cases is equal to 6 plus 12 that is equal to 18 and then x1 can take two values
So this will become 2 into this
So that is 2 into 18 that is equal to 36 cases that means if x 1 is equal to 1 3 then total number of cases is 36. now second part can be if x one is equal to two then x two plus x three must be odd now for odd how many cases will be there
So that is
So x two you can have odd and x three even because sum of r plus even is odd now x two odd you can have one three five
So there are three cases x three even there are two four six three cases
So total nine cases similarly you can have x two even and x three odd
So x two even there are two four that is two cases and x three r there are one three five and seven four cases
So there are total eight cases
So in this case you are getting total number of cases is equal to
So in this case total number of cases is equal to 9 plus 8 that is equal to 17
So total number of cases in which x 1 plus x 2 plus x 3 is odd is equal to 36 plus 17 that is equal to 53
So the probability is 53 divided by 1 0 5. let us look at the second case what is the probability that x1 x2 x3 are in an arithmetic progression
So for arithmetic progression you have to see that x1 x2 x3 are chosen in such a way that x1 minus x2 and x2 minus x3 they are same that is the common difference is there now what are the possibilities of the common differences
So let d be the common difference let us see what are the ways in which this can be done
So for example if i choose x 1 is equal to 1 then x 2 and x 3 can be 1 1 each
So that is 0 common difference or you can have 1 2 3
So that is one common difference you can have 2 2 2 2 3 but then 4 is not possible then you can have 3 but then 3 3 3 is possible 3 4 and
So on is not possible you can have 3 2 1 that is the minus 1 as the difference or you can have 1 3 5 that is possible or you can have 2 4 6 or you can have 3 5 seven
So let us look at all these combinations
So d can take values minus one zero one two and three then what are the number of cases ah i told you for minus one the only possibility is that x one should be three x two

should be two and x three should be one
So the total number of cases is only one what are the possibility for 0
So 0 is possible when x 1 x 2 x 3 are equal that means you can have 1 1 1 2 2 2 or 3 3 3
So there are three such cases common difference one that is possible if you have one two three two three four and three four five
So there are total number of three cases again let us look at the common difference two
So common difference two can be possible if you have x one x two x three as one three five two four six r three five seven
So total number of cases is three then the common difference 3 is possible if i have 1 4 7
So there is only one possibility because 2 5 and then 8 is not there
So this is the total number of cases total number of cases is eleven
So the required probability that the probability that x 1 plus x 2 plus x 3 that is x 1 x 2 x 3 are in an arithmetic progression is 11 by 1 0 5
So such kind of problems are usually asked in the competitive exams i will spend some more time on solving j type of questions in the next class and i will spend few lectures on the discrete distributions
So please follow the next few lectures also because i will be solving many interesting problems which are asked commonly in the entrance examinations thank you you