

so in the previous lectures i have discussed in detail the concepts of probability various rules for evaluation of the probability and the concept of independence random variables discrete distributions and binomial distribution in this lecture i will solve various problems which are asked frequently in various entrance examinations for example engineering entrance examinations are examinations of some other universities etcetera this will cover almost all the topics that we have done till now i will again advise the students please revise your concepts of permutations and combination because in counting many times in the probability problems we use these concepts a four digit number is chosen at random find the probability that there are exactly two zeros in that number ah one may wonder that whether these problems are for academic interest only or whether this will have any practical use also

so some of you might have heard the name of codes or cryptography etc

so in that to design the codes or to break the code this kind of problems are faced and therefore calculating probabilities of various possibilities is certainly one of the problems there okay

so if we have to choose a four digit number what is the total number of such numbers

so the total number of four digit numbers in the first place we can have any number between one to nine

so there are total nine possibilities and in the second third and fourth place zero can also be there

so total ten possibilities at each place

so that is total nine thousand such cases are there now if there have to be two zeros in that number if we have to get two zeros then let us choose the places first place numbers one to nine

so you can have nine such possibilities and in another place you can have numbers 1 to 9

so nine such cases are there two places we are fixing as zero

so there is no choice there however the first place cannot be zero out of the remaining two places remaining three places two places are zeros

so those places can be determined in three c two ways the two places where zero can be placed can be chosen in three c two that is three ways okay

so we have counted the possibility of having a four digit number in which two zeros are there

so the total number of ways is total number of favorable cases let us write it here it will be  $9 \times 9 \times 3$

so the probability that number has two zeros the favorable number of cases is  $9 \times 9 \times 3$  and the total number of cases is 9000

so after simplification it becomes simply  $\frac{27}{1000}$  or you can say 0.027 okay

let us take another problem in which the set theoretic notations are used let e and f be independent and probability of e plus probability of f is equal to one also let probability of e intersection f be equal to  $\frac{2}{9}$  and it is given that probability of e is greater than probability of f then you have to find probability of e

so let us assume probability of e is equal to say p now it is given that probability of e plus probability of f is equal to one then this implies that probability of f will become equal to one minus p now probability of e intersection f because e and f are independent this becomes probability of e into probability of f

so that is equal to p into 1 minus p that is  $\frac{2}{9}$  as you can see this is simply a quadratic equation

so p can be  $\frac{1}{3}$  or  $\frac{2}{3}$  because for these two values only this equation

will be satisfied now if i choose p to be 1 by 3 then probability of f will become 2 by 3 but it is given that probability of e is greater than probability of f

so we will choose probability of e to be 2 by 3 because in that case probability of f will become 1 by 3.

in this problem i have used the concept of independence of events and a direct solution of the system of a non-linear equation let me give another example in which the set theoretic probabilities are used let e and f be any two events with  $0 < \text{probability of e} < 1$  and  $0 < \text{probability of f} < 1$  and also it is given that probability of e is less than probability of e given f that is the conditional probability of e given f is more than probability of e and under these conditions we want to prove certain statements probability of f is less than probability of f given e probability of e is greater than probability of e given f complement probability of f is greater than probability of f given e complement now it is given that probability of e is less than probability of e given f that is the condition given there if we simplify this condition and apply the definition of conditional probability probability of e given f is probability of e intersection f divided by probability of f

so this means probability of e intersection f is greater than probability of e into probability of f now this you can write as probability of e intersection f divided by probability of e greater than probability of f now this is nothing but probability of f given e greater than probability of f

so this is proving the statement 1.

in the statement one we were supposed to prove probability of f is less than probability of f given e which is established here now let us take the second one if we use this equation let me call it one we can consider probability of f minus probability of e intersection f it will become less than probability of f minus probability of e into probability of f

so this means the left hand side becomes probability of f intersection e complement and the right hand side becomes probability of f 1 minus probability of e

so this is equivalent to statement that probability of f intersection e complement is less than probability of f into probability of e complement

so this if you write it becomes probability of f intersection e complement divided by probability of e complement less than probability of f

so this statement is equivalent to probability of f given e complement is less than probability of f

so this is the statement number 3 probability of f is greater than probability of f given e complement

so what we have done we have used the given condition that is probability of e is less than probability e given f which we simplify to probability of e intersection f greater than probability into probability f in this i did a little bit of manipulation on both the sides i did p f minus

so the inequality gets reversed after simplification of this we get the required result

so actually we have proved the third one here let us look at the second one again to prove the second one i will use this one again from one if we consider probability of e minus probability of e intersection f then that is less than probability of e minus probability of e into probability of f

so that means probability of e intersection f complement less than probability of e into 1 minus probability of f

so that is implying that probability of e intersection f complement is less than probability of e into probability of f complement that means probability of

$P(e \cap f^c) / P(f^c)$  is less than  $P(e)$  which is equivalent to saying  $P(e | f^c) < P(e)$  which was the statement to be proved in 2 that is  $P(e) > P(e | f^c)$

so we have established that statement here once again in this example we have used the concept of conditional probability

so we have actually applied the definition of conditional probability then we have actually used the addition rule at two places for example  $P(e \cap f) + P(e \cap f^c) = P(e)$  that is used here similarly in this i have used  $P(e \cap f) + P(e \cap f^c) = P(e)$

so this is the addition rule if you make the diagram like this i have two sets  $e$  and  $f$  here then  $e \cap f^c$  will be this and  $e \cap f$  will be this

so the union of this is  $e$  similarly if i consider  $f \cap e^c$  then that is this portion  $f \cap e^c$  is this

so if i take the union of the two i will get  $f$  let me give one example in which some sort of geometric argument is used

so two points are chosen at random on a line segment of length 9 centimeter so there is a line segment find the probability that the distance between these two points is less than 3 centimeter now let us look at this geometrically

so there is a line segment of 9 centimeter we choose  $x$  and  $y$  here ok and it can be here also like  $x$  can be here  $y$  can be here  $x$  can be here  $y$  that means  $x$  can be less than  $y$   $x$  can be greater than  $y$  but we want this distance to be less than 3 centimeter

so a better way of looking at it can be we can plot it in a two dimensional plane and suppose this is  $x$  axis this is  $y$  axis and we consider a square of size nine that means on this side you have 9 9 on this side also and this is the line so we want  $x - y < 3$  or  $y - x < 3$ .

so if we consider these two lines here that is  $x - y = 3$  this is the line  $x - y = 3$  and we consider another line  $x - y = -3$  ah if you want you can just ah check the condition if  $y = 0$   $x = 3$  if you consider  $x = 0$  then  $y = -3$

so if you join the points and you draw it you will get this similarly on this side if you say if  $x = 0$   $y = 3$  if  $y = 0$   $x = -3$ .

so this is the line that you get here

so actually we are considering  $x, y$  in the two dimensional plane here we can consider the point  $(x, y)$  in a square in a two dimensional plane

so this entire area is 9 into 9 that is 81 square centimeter and the shaded area is the required area the required probability is equal to area of the shaded region divided by total area ah this can be done in a easier way also like you can have this right angle triangle

so the area of this will be  $6 \times 6 \times 2$  that is 36 and the same thing here

so  $36 + 36 = 72$

so  $72 / 81$  that is equal to  $8/9$ .

so this is an application where we are using direct geometric argument although one can use a certain bivariate distribution and do use a little bit of advance probability theory but here i am showing that by a simple geometric argument we

can get the required probability here let  $u_1$  and  $u_2$  be 2 urns such that  $u_1$  contains three white and two red balls and  $u_2$  contains only one white ball a fair coin is tossed if head appears then one ball is drawn at random from  $u_1$  and put in  $u_2$  however if tail appears then two balls are drawn at random from  $u_1$  and put into  $u_2$  now one ball is drawn from  $u_2$  at random that means actually  $u_2$  may have either two balls or it may have three balls from that we are selecting one ball at random what is the probability that the ball drawn from  $u_2$  is white given that the ball drawn from  $u_2$  is white what is the probability that head appeared on the coin ah this is one of the problems which is asked in one of the earlier engineering entrance examinations let us look at the solution in full so i am using the set theoretic probability here corresponding to each event we will define a set

so let  $a$  denote the event that ball from  $u_2$  is white let  $b_1$  be the event that there is head on the coin and  $b_2$  be the event that there is tail on the coin

so coin we are assuming fair

so the probabilities of  $b_1$  and probability of  $b_2$  will be equal to half now what is the probability of say  $a$  given  $b_1$

so  $a$  given  $b_1$  means if head comes then we are drawing one ball from  $u_1$  and putting into  $u_2$  now that ball can be a white ball or that ball can be a red ball depending upon that we are asking what is the probability of drawing a white ball from the urn  $u_2$

so let us write this event properly white ball is drawn from  $u_2$  given that white ball is drawn from  $u_1$  into probability that white ball is drawn from  $u_1$  plus probability that white ball is drawn from  $u_2$  given that red ball is drawn from  $u_1$  into probability that red ball is drawn from  $u_1$

so what we have done we have applied the theorem of total probability when we are drawing a ball from  $u_1$  now  $u_1$  contains three white and two red balls

so the ball can be white or it can be red if the ball drawn is white then on  $u_1$  will have on  $u_2$  will have both the white balls

so the probability of drawing a white ball will become simply one but what is the probability that a white ball is drawn from  $u_1$  it will become simply three by five because there are total five balls in  $u_1$  out of which three are white

so the probability will become three by five if a red ball is drawn and put into  $u_2$  then the probability of drawing a white ball will become half because on two then we will have one white and one red ball and the probability that a red ball is drawn from  $u_1$  will be two by five

so this is nothing but four by five similarly let us calculate probability of  $a$  given  $b_2$  probability of  $a$  given  $b_2$  that is probability of white ball is drawn from  $u_2$  given that now in the when a tail is there two balls are drawn from  $u_1$  and put it into  $u_2$

so both can be white one can be white one can be red or both can be red

so let us look at all the possibilities again by applying theorem of total probability

so two white balls are drawn from  $u_1$  into probability that two white balls are drawn from  $u_1$  plus probability that white ball is drawn from  $u_2$  given that one white and one red ball is drawn from  $u_1$  into probability that one white and one ball is drawn from  $u_1$  plus probability that white ball is drawn from  $u_2$  given that two red balls are drawn from  $u_1$  into probability that two red balls are drawn from  $u_1$  let me repeat the sentence here we have applied the theorem of total probability  $b_2$  means that a tail was obtained when a coin was tossed if a tail is obtained then we are drawing two balls from on  $u_1$  and putting into  $u_2$

so we are looking at three possibilities both the balls from  $u_1$  can be white one ball can be white or one can be red or both balls can be red

so we have just applied the theorem of total probability for describing the probability of white ball is drawn from  $u_2$  given that a tail is obtained

so now we completely describe this by looking at these possibilities if two white balls are drawn from  $u_1$  then on  $u_2$  will have all the white balls

so the probability of drawing a white ball will be simply one however what is the probability of drawing two white balls from on  $u_1$  because there are three white balls

so three  $C_2$  divided by total five are there five  $C_2$  plus in the case one white and one red is drawn from  $u_1$  and put it into  $u_2$  then  $u_2$  will have two white and one red

so drawing a white ball the probability will become two by and the probability of this possibility will be  $3 C_1 2 C_1$  divided by  $5 C_2$  plus the next one is two red balls are drawn from  $u_1$  and put into  $u_2$  then in  $u_2$  there is one white and two red balls

so the probability of drawing one white ball will become one by three and the probability of this choice will be  $2 C_2$  divided by  $5 C_2$  now these expressions one can easily simplify  $3 C_2$  is  $\frac{3 \cdot 2}{5 \cdot 2}$  is  $\frac{10}{30}$   $3 C_1$  is  $\frac{3 \cdot 2 \cdot 1}{5 \cdot 2 \cdot 1}$  is  $\frac{6}{10}$   $2 C_2$  is  $\frac{2 \cdot 1}{5 \cdot 2}$  is  $\frac{2}{10}$

so after simplification this value becomes  $\frac{11}{15}$ .

now you see we are asked to calculate probability that the ball drawn from  $u_2$  is white this is the event  $a$  now we have calculated probability of  $a$  given  $b_1$  and probability of  $a$  given  $b_2$

so we apply again the theorem of total probability probability of  $a$  is equal to probability of  $a$  given  $b_1$  into probability of  $b_1$  plus probability of  $a$  given  $b_2$  into probability of  $b_2$  that is  $\frac{4}{5}$  into  $\frac{1}{2}$  plus  $\frac{11}{15}$  into  $\frac{1}{2}$

so that is equal to  $\frac{23}{30}$

so the probability of drawing a white ball from on  $u_2$  is  $\frac{23}{30}$ .

now we are asked to calculate the probability that if the ball drawn from  $u_2$  is white what is the probability that head appear on the coin that is the probability of  $b_1$  given  $a$

so we have to use the base theorem here to find probability of  $b_1$  given  $a$  we use base theorem

so probability of  $b_1$  given  $a$  that is equal to probability of  $a$  given  $b_1$  into probability of  $b_1$  divided by probability of  $a$  given  $b_1$  into probability of  $b_1$  plus probability of  $a$  given  $b_2$  into probability of  $b_2$ .

so if we substitute all the values here that is  $\frac{4}{5}$  into  $\frac{1}{2}$  divided by  $\frac{23}{30}$  that is equal to  $\frac{12}{23}$  again let me tell you we have used the concept of the theorem of total probability three times in this problem firstly to calculate the probability of drawing a white ball from on  $u_2$  when a head is observed

so here there were two possibilities that there can be a white ball from on  $u_1$  which is put into  $u_2$  or there can be a red ball from  $u_1$  which is put into  $u_2$  in the second case to calculate the probability of  $a$  that is white ball is drawn from on  $u_2$  when a tail is observed then there are three cases because in that case we are drawing two balls from on  $u_1$

so both can be white  $r_1$  can be white one can be red  $r$  both can be red

so depending upon this we have calculated various probabilities in the second part of this we have used the bayes theorem here ah let me solve another problem which is of similar type appearing in an entrance examination there are  $n$  arms numbered  $1, 2, \dots, n$  each containing  $n + 1$  balls iron eye contains eye white balls and  $n + 1 - i$  red balls  $i$  is equal to  $1$  to  $n$  and arm  $i$  is selected and a ball is drawn from it let  $u_i$  denote the event that on  $i$  is selected and let  $w$  be the event that a white ball is drawn from the selected arm further

suppose that  $e$  denotes the event that an even numbered one is selected let probability of  $u_i$  be proportional to  $i$  for  $i$  is equal to 1 to  $n$  then you have to find limit of probability  $w$  as  $n$  tends to infinity in the second case if probability of  $u_i$  is constant for  $i$  is equal to 1 to  $n$  where  $c$  is a constant find probability of  $u_n$  given  $w$  third if probability of  $u_i$  is equal to  $1/n$  for  $i$  is equal to 1 to  $n$  and  $n$  is an even positive integer find probability of  $w$  given  $e$

so let me just repeat the problem once again there are  $n$  urns which we identify as 1 to  $n$

so some numbering is done there each of them has  $n + 1$  balls in the  $i$  urn there are  $i$  white and  $n + 1 - i$  red balls now one urn is selected at random and a ball is drawn from it now we are identifying certain events

so  $u_i$  is the event that urn  $i$  is selected and  $w$  is the event that a white ball is drawn from the selected urn and  $e$  is the event that an even numbered urn is selected based on this we are asking some problems for example if probability of  $u_i$  is proportional to  $i$  then what is probability of  $w$  find the limit as  $n$  tends to infinity similarly if probability of  $u_i$  is constant then find probability of  $u_n$  given  $w$  and in another one find probability of  $w$  given  $e$

so let us look at the solution of this if probability of  $u_i$  is proportional to  $i$  then we can write probability of  $u_i$  is equal to  $k \cdot i$  for  $i$  is equal to 1 to  $n$  now the sum of all probabilities must be equal to one because one of the urns has to be chosen

so it gives you  $k \sum_{i=1}^n i$  is equal to one to  $n$  now that is equal to  $k$  times  $n$  into  $n + 1$  by 2 that is equal to 1

so  $k$  becomes  $2$  divided by  $n$  into  $n + 1$

so probability of  $u_i$  is equal to  $2i$  divided by  $n$  into  $n + 1$  for  $i$  is equal to 1 to  $n$  that means the urn 1 the first one is selected with probability  $2$  divided by  $n$  into  $n + 1$  the urn 2 is selected with the probability  $4$  by  $n$  into  $n + 1$  the urn  $n$  is selected with the probability  $2n$  divided by  $n$  plus 1

so the question that is posed here is what is the probability of  $w$  that is a white ball is drawn from the selected urn and

so we can apply the theorem of total probability that will give probability of  $w$  is equal to  $\sum_{i=1}^n$  probability of  $w$  given  $u_i$  into probability of  $u_i$   $i$  is equal to 1 to  $n$

so that is equal to in the  $i$ -th urn there are  $i$  white balls

so the probability of choosing a white ball from the  $i$ -th urn is  $i$  divided by  $n$  plus 1 and probability of  $u_i$  we have just now calculated it is  $2i$  divided by  $n$  into  $n + 1$   $i$  is equal to 1 to  $n$

so this you can see it is  $2$  divided by  $n$  into  $n + 1$  square summation  $i^2$   $i$  is equal to 1 to  $n$  that is the sum of squares of first  $n$  positive integers the formula is known that is  $n$  into  $n + 1$  into  $2n + 1$  by 6.

so if we apply that we get  $2n$  into  $n + 1$  square into  $n$  into  $n + 1$  into  $2n + 1$  by 6.

so we can easily simplify this it is equal to twice  $2n + 1$  divided by 6 into  $n + 1$  because these terms  $n$  cancels out  $n + 1$  cancels out so we get this

so if I take the limit of this as  $n$  tends to infinity you will get simply  $2$  into  $2$  by 6 that is equal to  $2$  by 3 as  $n$  tends to infinity  $1/n$  goes to 0

so if you divide by  $n$  here you get  $2 + 1/n$  and here you get  $1 + 1/n$  so the limit is  $2$  into  $2$  by 6 that is  $2$  by 3.

in the second part probability of  $u_i$  is equal to constant

so if the probability is constant then  $c \cdot n$  is equal to one that means it

must be equal to  $\frac{1}{n}$

so in the second part probability of  $u_i$  will be equal to  $\frac{1}{n}$  for  $i$  is equal to 1 to  $n$  now you are asked to calculate probability of  $u_n$  given  $w$

so we can apply Bayes theorem here then it becomes probability of  $w$  given  $u_n$  into probability of  $u_n$  given  $w$  into probability of  $u_i$  is equal to one to  $n$  in the  $n$ th iron there are  $n$  white balls

so the probability will be  $\frac{n}{n+1}$  and these are all one by  $n$  divided by  $\sum_{i=1}^n$  in the  $i$ th you have  $i$  white ball

so the probability will be  $\frac{i}{n+1}$  and the probability of  $u_i$  is  $\frac{1}{n}$  is equal to  $\frac{1}{n}$

so this becomes simply  $n$  divided by  $\sum_{i=1}^n i$  that is  $n$  divided by  $n+1$  by 2 that is equal to  $\frac{2}{n+1}$

so the probability that the  $n$ th iron was chosen given that there is a white ball is  $\frac{2}{n+1}$ .

now in the third part what is the probability of  $e$  what is the event  $e$  that an even numbered iron is selected

so that will become equal to  $\sum_{i=1}^m$  probability of  $u_{2i}$  is equal to  $\frac{1}{m}$  if  $i$  assume  $n$  is equal to  $2n$

so that is becoming  $m$  divided by  $2m$  that is half

so if we consider probability of  $w$  intersection  $e$  then that is equal to  $\sum_{i=1}^m$  probability of  $w$  intersection  $u_{2i}$  is equal to  $\frac{1}{m}$

so again we can apply the multiplication rule probability of  $w$  given  $u_{2i}$  into probability of  $u_{2i}$  is equal to one to  $n$  in the  $2i$ th iron there are  $2i$  white balls

so that becomes  $\frac{2i}{2m+1}$  the probability of  $u_{2i}$  is one by  $2m$  is equal to  $\frac{1}{m}$

so this is simply the sum of first  $m$  number

so it becomes  $\frac{1}{m} \sum_{i=1}^m 2i$  that is  $\frac{m(m+1)}{2}$

so this is easily simplified it is  $\frac{m+1}{2}$

so if  $i$  calculate probability of  $w$  given  $e$  that is equal to probability of  $w$  intersection  $e$  divided by probability of  $e$  then that is equal to  $\frac{m+1}{2m+1}$  that is equal to  $\frac{n+2}{2n+1}$   $i$  have put  $n$  is equal to  $2m$  here there is another problem an experiment has 10 equally likely outcomes let  $a$  and  $b$  be two non-empty events of the experiment let  $a$  have four elements that means out of these 10 equally likely outcomes 4 belong to  $a$  they are favorable to  $a$  if  $a$  and  $b$  are independent how many elements can be in  $b$

so if  $i$  use the notation  $n_e$  to be the number of elements in  $e$  then we are having  $n_s$  that is the number of elements in the sample space is 10 and  $n_a$  is given to be four now  $a$  and  $b$  are independent

so probability of  $a$  intersection  $b$  will be equal to probability of  $a$  into probability of  $b$  since the items are equally likely the outcomes

so therefore we can apply the classical definition  $n(a \cap b)$  divided by  $n(s)$  that is equal to  $\frac{n(a)}{n(s)} \cdot \frac{n(b)}{n(s)}$  we can substitute the values here  $n(a \cap b)$

so this  $2n_s$  are there into  $n(s)$  is equal to  $n(a) \cdot n(b)$  now  $n(s)$  is 10 this is 4.

so if  $i$  take  $n(a \cap b)$  to be 2 and  $n(b)$  to be 5 then the two sides are equal similarly if  $i$  take  $n(b)$  equal to 10 itself that means all elements are there then  $n(a \cap b)$  will become 4 because  $a$  has 4 elements

so this will become 40 and this side will become 40.

so only possible values of  $n \cap b$  can be 5 or 10 if  $n \cap a$  is 5 then  $n \cap b$  must be equal to 2 and if  $n \cap a$  is equal to 10 then  $n \cap b$  is equal to  $n \cap a$  that is equal to 4.

in this problem the peculiar thing that you note here is that we have actually used the number of terms or you can say favorable number of outcomes to an event here that is explicitly used here although in many problems we have done that that is we have calculated the favorable number of cases but in this particular problem we have used an explicit notation for that and that is used for solving the problems

so in this particular course i have devoted enough time on explaining the basic concepts of probability which includes the conditional probability base theorem the theorem of total probability the concept of independence as well as we have also briefly touched upon the concept of random variables out of that we have devoted some time to the discrete random variables especially the binomial distribution and we have also looked at the concept of the mean or the average value or the expectation the variability of the distribution in terms of variance and the standard deviation in order to properly do justice to this part it will be better if you also have done the permutations and combination because in some of the problems they have been used you