i have discussed in detail the basic definitions of probability and certain rules based on the basic definitions ah they are addition rule a generalization of that called the generalize general addition rule we studied the concept of conditional probability multiplication rule ah the theorem of total probability bayes theorem and concept of independent events now today i will devote my time to solving some of the problems of probability ah in your school books you might have seen that there are many problems

so i will solve many problems which are of similar nature and few problems which might have appeared in some of the competitive exams like joint entrance examination etcetera

so let me start with some of the problems and these are basically applications of all the ah theorems and formula that we have studied till now

so let me start

so let a and b any two events and some conditions are given probability of a complement is given to be point three probability of b is given to be point four probability of a minus b is given to be point five and we are required to find probability of b given a union b complement

so we are asking a conditional probability of b given a union b complement

so let us see by applying the formula for the conditional probability what is the expression for the probability of b given a union b complement if you remember the formula of probability of e given f it is probability of e intersection f divided by probability of f

so this becomes probability of b intersection a union b complement divided by probability of a union b complement let us look at the numerator here

so let us simplify this that is probability of b intersection union of two events

so we can apply the distributive property of unions and intersections

so this becomes b intersection a union b intersection b complement

so we get it as b intersection a now b intersection b complement is nothing but empty set

so now you are saying this union phi

so it is nothing but probability of a intersection b

so that means the numerator probability is equivalent to probability of a intersection b now if we consider say a when diagram here and event a and event b then this is a intersection b what is given to us we are given probability of a complement now if probability of a complement is given you can easily find out probability of a probability of b is given what else is given probability of a minus b what is a minus b you can easily see that this part is the probability of a minus b which you can also say a intersection b complement therefore the set a is equivalent to a minus b union a intersection b that means we can write a as a minus b union a intersection b that means probability of a is equal to probability of a minus b plus probability of a intersection b that means probability of a intersection b is equal to probability of a minus probability of a minus b now we are given here probability of a compliment is point three

so probability of a will become one minus point three that is point seven

so this is equal to point seven minus probability of a minus b which is point five

so this value turns out to be point two that means probability of a intersection b that is this numerator quantity is 0.
2 ah now let us look at the denominator

so the denominator is probability of a union b complement on a and b complement we apply the addition rule

so this is equal to probability of a plus probability of b complement minus probability of a intersection b complement

so you can actually see that in the formula for the addition rule i have written the formula probability of a union b

so here b is replaced by b complement

so there is no difficulty in writing down this term

so once again you see what are the values that we are supposed to write here probability of a that is one minus probability of a compliment

so that is equal to point seven plus probability of b complement that is one minus probability of b

so that is one minus point four that is point six now what is probability of a intersection b complement now from the venn diagram you can see b complement is this entire portion and a the intersection of b complement with a is exactly this portion which i wrote as a minus b

so this a minus b and a intersection b complement they are the same

so that is equal to point five

so you can evaluate it this is equal to point eight

so in the expression for probability of b given a union b complement the numerator is probability of a intersection b which is point two and the denominator that is probability of a union b complement is point eight

so let us substitute here to get the value probability of b given a union b complement as point two divided by point eight that is equal to one by four or point two five now you see what are the principles of the probability that i have used in this problem first is the definition of the conditional probability then in resolving the numerator i have used the distributive laws of set theory and then we get empty set etcetera to calculate probability of a intersection b again i have used the additive property of the probability because i needed probability of a intersection b which is not given here but whatever is given here i just saw through this diagram that we are given what is probability of a and we are given what is the probability of a minus b

so from here we can easily calculate the probability of a intersection b in this particular fashion

so this value is coming out to calculate the denominator i use the addition rule and this gives you the value 0.
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so the purpose of solving this problem was to show that if the probabilities of certain events are given then using that we can evaluate probabilities of various related events ah let us take a similar problem again let a and b be any two events with probability of a union b is equal to point seven probability of a is equal to point five and probability of ah b is equal to say point three find probability of a given b complement let us look at the solution we are required to find probability of a given b complement if we apply the conditional probability then it is probability of a intersection b complement divided by probability of b complement ah again you can see probability of b complement is available from probability of b ah let us look at this numerator here now numerator is probability of a intersection b complement again through a venn diagram we can easily understand if this event is a this event is b then a intersection b complement is this part which is actually a minus b we are given a union b a and b

so if we can look at a intersection b we can calculate using the from the addition rule probability of a union b is equal to probability of a plus probability of b minus probability of a intersection b

so that gives us probability of a intersection b is equal to probability of a plus probability of b minus probability of a union b ah i have written this form because the values of probability a b and a union b are available to us

so we substitute it here and get point five plus point three minus point seven

which is equal to point one

so the probability of a intersection b is equal to point one that is this part

so now since the probability of a is given to me we can calculate the probability of a intersection b complement by taking the difference

so we use the another formula now that is the probability of a is equal to probability of a intersection b plus probability of a intersection b complement that is the probability of a intersection b complement is equal to probability of a minus probability of a intersection b now in this one you can see point five is the probability of a and probability of a intersection b we calculated to be point one

so this is equal to point four

so if we substitute probability of a given b complement we calculated it to be probability of a intersection b complement divided by probability of b complement

so we substitute the values 0.
7 divided by probability of b complement is point seven

so that is point

so that is equal to four by seven again you can see that in this particular problem i have used ah the formula for the conditional probability and in order to evaluate the terms of the involved in the ratio we have used the addition rule because the event a we have split as a intersection b and as a intersection b complement and then for b complement we have directly calculated let us take one more example let a and b be two events with probability of a is equal to point five probability of a intersection b is equal to point two and probability of a given b complement is equal to point four we want to find what is the probability of b

so we will try to use the equations here probability of a given b complement that is equal to probability of a intersection b complement divided by probability of b complement we have just now seen that the probability of a intersection b complement is nothing but probability a minus probability of a intersection b

so these values are given to us we will utilize them and probability of b complement is one minus probability of b

so this equation then becomes the left hand side is point four is equal to probability a is point five minus point two that is probability of a intersection b divided by one minus probability of b

so this is a very simple equation and we can easily solve it

so we get one minus probability of b is equal to three by four which implies that probability of b is equal to one by four

so basically in these problems you have seen that using the basic rules of probability we can evaluate probabilities of certain other events let us continue with some more problems and earn has four coins and it is given that three coins are fair and one coin is biased then the probability of head for the biased coin is three by four

so that means three coins are fair

so here the probability of head and tail is half each and for the bias coin the probability of head is three by four

so the probability of tail will be one by four a coin is drawn at random from the urn and tossed what is the probability that a head is observed if a head is observed what is the probability that the biased coin was drawn here you can see that the problem is in two steps firstly a coin is drawn and after that the coin is tossed

so naturally the coin can be a fair coin or it can be a biased coin and therefore the probability of head will be dependent upon whether the fair coin

was drawn or the biased coin was drawn

so this is a perfect setting for application of the theorem of total probability

so let us look at it

so i define some events let h be the event that a head is observed and i also define events let e be the event that a biased coin was drawn and f is the event that a fair coin is drawn then what is the probability of h given e h given e means that from the on we draw a biased coin and then we toss it then the probability of head will be three by four what is the probability of head if a fair coin is drawn then the probability is half also what are the probabilities of e and f now there are four coins out of which only one coin is biased

so if we are drawing at random the probability that we draw a biased coin will become one by four and the probability that we draw a fair coin it will become three by four

so we have evaluated basic probabilities which are related to this problem now what is asked we are asked what is the probability that a head is observed

so head is observed means probability of h

so we want to find out what is the probability of h

so the probability of h is now if we apply the theorem of total probability then it is h given e into probability of e plus probability of h given f into probability of f all the terms which are given in this equation they are available to us probability of h given is three by four probability of e is one by four probability of h given f is half and the probability of f is three by four

so we can just calculate these values it is equal to ah nine by sixteen

so the probability of head ultimately is nine by sixteen let us look at the second part of the problem the second part of the problem was what is the probability that the biased coin was drawn if a head is observed that means it is the reverse because we are asking the probability of the event which appeared first

so because coin is drawn first

so we now know the final outcome

so what is the probability of e given h

so by bayes theorem it is h given e into probability of e divided by probability of h ah

so all these terms are available three by four and one by four divided by nine by sixteen

so it is equal to one by three ah let us try to review this what is the probability of e the probability of e is one by four ok that means ah since there are four coins out of which only one coin is biased

so the probability of drawing the biased coin was one by four now if the outcome is known that it is head then the probability of biased coin has increased from one by four it has become one by three the reason is that there is more chance of head from the biased coin therefore the probability gets revised

so this is actually the advantage of the base theorem that means by knowing the final outcome we are able to revise the probabilities of the prior events the events which actually occur before and as i had mentioned to you earlier that this is ah very useful when we want to find out the causes of accidents some mishaps in the factories ah some forensic investigations etcetera in all these places the bayes theorem plays an important role because we can actually revise the probabilities of the prior events dependent upon what is the final outcome which is known to us let us continue and look at some other applications ah a pair of dice is rolled until a sum of seven appears

so ah if the dice are tossed then both the dice have some upper face value say two and three etcetera three and four three and three four and two etcetera
so we look at the sum
so that sum is recorded
so you when the sum saving is observed then we stop
so we want to find out the probability find the probability that the sum 7 appears first on the third roll ok ah first of all if we toss a pair of dice once
so we assume fair let me write it here fair dice
so that the probability of each outcome is equal
so there are total 36 cases now out of those 36 cases which cases lead to the sum 7.

so let us look at the possibilities here if we say sum is the seven then we can have one six and of course six one then we can have two five and five two and then three four and four three
so if we consider this event as e that the sum is seven then it is consisting of six elements
so if we look at the probability of e that is equal to six by thirty six that is one by six let me call it ah probability as p now the next question is that what is the probability that the sum of seven appears first on the third roll of the pair of fair dice what it means it means that on the first roll we do not get the sum seven that means event e compliment happens similarly in the second toss also the event e complement occurs then in the third the event e occurs now again ah here we make the assumption that the tossings are independent that means whenever we toss and we record then second time the tossing is independent of the previous toss therefore we can actually apply the multiplication of the probabilities if you remember the definition of independence of events we say events a b are independent if probability of a intersection b is equal to probability of a into probability of b if we have three then we have several conditions but all of them mean that probability of the intersections is equal to the product of the probabilities
so we apply this now
so probability of sum appears first on the third roll
so that is equal to
so we can say e compliment occurs on first roll then intersection e complement occurs on second roll intersection e occurs on third row
so i have written the event sum appears first on the third row as intersection of three events
so e complements that seven does not occur on the first roll seven does not occur on the second row and it occurs on the third row it is intersection of the three events now because of the independence of the trials this becomes probability of e complement on the first roll into probability of e complement on the second roll into probability of e now in each role the probability of e and e complement they remain the same ah
so we calculated the probability of e as equal to one by six
so the probability of e compliment becomes five by six then again five by six and then probability of e that is one by six
so we get twenty five by two one six
so if we analyze it carefully what are the principles of probability that have been used here firstly we have used the mathematical definition of probability or the classical definition because we have considered fair dice
so the cases are all equally likely all the 36 cases are equally likely secondly we have used the independence concept that means the roles first second

and third they are independent therefore the probabilities can be multiplied ah let us look at some further applications of this rules of probability

so let a and b be two events which are independent with say probability of a minus b is equal to three by twenty five and probability of b minus a is equal to eight by twenty five further probability of a is given to be greater than half then you have to find what is the probability of b from the given information we have to extract the probabilities of a and b and ultimately we are asked to find probability of b additional information that is given to us is that events a and b are independent let us see first of all through a vein diagram that what are the probabilities that are available to us and how we can utilize them

so suppose event a is here and event b is here then a minus b is this term
so this probability is given to be three by twenty five and similarly b minus a becomes this

so these are the values that are available to us in order to make use of independence i need probabilities which have intersections available to us
so what is given to us probability of a minus b now a minus b as you can see it is also probability of a intersection b complement you can see from here if i take b compliment here then that is a external set and taking intersection with a will give me this region similarly if i look at b minus a that is equal to probability of b intersection a complement

so these values are given to us that is equal to three by twenty five and this is given to be eight by twenty five ah now we will utilize the condition of independence

so first of all i will prove that if a and b are independent then a and b complement is independent and b and a complement are also independent because then i can write this as the product of the probability of a and probability of b complement and here i can write probability of b into probability of a complement

so let us first prove this first we prove that if a and b are independent then a and b complement are also independent ok
so if i consider probability of a intersection b complement now we have already seen that probability of a intersection b complement can be written as probability of a minus probability of a intersection b ah in our one of the earlier problems we have used this thing

so i am not proving it again sorry a intersection b
so this is equal to probability of a minus because a and b are independent this can be written as probability of a into probability of b

so here you can take probability of a into 1 minus probability of b which is equal to probability of a into probability of b complement
so probability of a intersection b complement is equal to probability of a into probability of b complement which is the condition for independence ah similarly we can prove that a compliment and b are independent basically what we are saying is that if two events are independent then if i take complement of one of them then they remain independent if i take compliment of both of them they also remain independent if i just extend this argument on the next set of events

so this means that we are having probability of a into probability of b complement is equal to three by twenty five let us use some notation let us write say p is equal to probability of a and q is equal to probability of b then this implies p into one minus q is equal to three by twenty five similarly probability of b intersection a complement that will become probability of b into probability of a complement that is equal to one minus p into q that is equal to eight by twenty five

so i have ah two equations and two unknowns p minus p q is three by twenty five

and q minus p q is equal to eight by twenty five

so if i do some manipulation here in fact i subtract equation one from equation two then p q p q will cancel out and you will get

so let me write two minus one gives q minus p is equal to one by five that is eight by twenty five minus three by twenty five that is five by twenty five that is one by five

so let me call it equation number three now from one of these equations suppose from this q value i put here

so substitute using q from three in say two we get one minus p into p plus one by five is equal to eight by twenty five ah this is a simple ah quadratic equation you can actually write it it is equal to p square

so if i consider this minus p square plus p minus p by five plus one by five is equal to eight by twenty five

so we can further simplify this is minus p square plus four p by five plus ah if i bring it to the left hand side i get minus three by twenty five is equal to zero which is same as saying one by five minus p into ah three by five minus p or you can say with a minus sign here is equal to zero

so this means p is equal to one by five r three by five ah let us look at the condition given in the problem here it is given that probability of a is greater than half

so since i got two values one by five and three by five i will go for the value three by five here

so we can utilize here the second option since p is greater than half we take probability of a equal to three by five now if this is three by five and if i substitute in this i get q is equal to four by five

so this was our original intention to evaluate the probability of b

so the probability of b we are getting to be four by five here now you can see in this particular problem we have used the concept of independence and then the idea of complementation is used and then it is some something simple we are using simple ah mathematics where we are solving two equations in two unknowns of course the equations are not linear ah but they are easy to solve from substituting from one of them into another we get a simple quadratic and that quadratic can be solved and then using the another condition given in the problem we are able to finally get the values of the required probabilities

so let us look at some more problems here two fare dice are tossed together what is the probability that the product of the numbers is divisible by six

so when the two fair dice are tossed we look at the numbers observed on the upper faces and we multiply them

so what is the probability that this product is divisible by six

so there are total number of cases here is 36 the total number of outcomes is thirty six now out of this in which outcomes we have the product equal to which is divisible by six

so let us look at that

so let me define the event a that the product is divisible by six then what are the elements if the first one is one second is six if the first is two and second is three if the first is two second is six if the first is three and second is two if the first is three the second is four if the first is three the second is six if the first is four second is three if the first is four the second is six if the first is five and the second is six and if the first is six then whatever be the on the other one all of them are divisible by six

so six cases six plus six ah twelve and then you have two three three two and three four and four three

so six plus ah six twelve plus three

so we have fifteen cases

so probability of a is equal to fifteen by thirty six that is equal to five by twelve ah let me just ask another question in this problem ok in this problem what is the conditional probability that the sum of the numbers is at least 10 given that the product is divisible by six

so let us define the event b that the sum is at least ten then what are the elements in b you will have see if we have the first number as one two r three then no matter what we obtain on the second one the sum can never be ten

so the first number has to be at least four then the second can be six if the first one is five then you can have second one as five r six and if the first one is six then the second one can be four five r six

so if we want to calculate probability of a given b then we need probability of a intersection b divided by probability of b now how many elements are there in a intersection b

so let us look at this 4 6 is here 5 5 is not here because here this is not divisible by six five six six four six five six six

so out of this terms five terms are common to a

so the probability of a intersection b becomes five by thirty six and the probability of b is ah ok i have actually asked what is the probability of b given a

so it is b given a

so in the denominator i will have probability of a that is five by twelve

so that is equal to one by three ah you can see if i directly look at what is the probability of b it will be one by six because it is six by thirty six whereas the probability of b given a is one by three

so conditioning substantially changes this probability an experiment has two possible outcomes the first occurs with probability p square plus p by four and the second with probability three minus p by four what is the value of p

so i am saying the experiment has two possible outcomes

so the probabilities of the two outcomes are given their total sum must be equal to one

so we will have p square plus p by four plus three minus p by four is equal to one

so this will imply that p square is equal to one by four which will give me two values of p that is plus minus half if i take p is equal to plus half then the probabilities of two outcomes will be if i put p is equal to half this will give me one by four plus one by eight that is three by eight and the second value is three minus p by four

so that is three minus half that is five by two then divided by four

so it becomes five by eight

so three by eight plus five by eight is one if p is equal to minus half then the probabilities of two outcomes

so p square plus p by four

so this will become one by four minus one by eight that is one by eight and the second one is three minus minus plus

so that is three plus half that is seven by two by four that is seven by eight

so again you can see that the sum is equal to one

so now you see this problem is an illustration of the exhaustive nature of the outcomes that means if i have all the outcomes listed

so in this case i have made it simple we have only two possible outcomes now using that property we are able to solve a certain ah equation of course the equation becomes extremely simple it is simply p square is equal to one by four now what happens that usually when we say p is equal to plus minus half ah we will say that take only p is equal to half and ignore p is equal to minus half now that may lead to some sort of mistake because suppose there is a question

where multiple correct options are correct multiple options are correct in that case both the options are correct whereas if you ignore p is equal to minus half actually it is a valid option because this is also leading to correct values here for the two outcomes only thing is here you have three by eight and five by eight and in the second one you are having one by eight and seven by eight each of coefficients a b c in the system of linear equations a x plus b y is equal to 0 and b x plus c y is equal to zero this each coefficient is determined by tossing a fair die ok
 so that means if we toss once whatever number is there we call it a we dos it again whatever number is observed we call it b and we toss it again and whatever number is there we call it c what is the probability that the system has non trivial solutions ah ok what is the meaning of non trivial solution see if we look at homogeneous system of linear equation then if you put x equal to zero and y is equal to zero then it is always a solution
 so we are looking at the solution which is having values not zero
 so ah in the system of linear equation you might have done the cramer rule the condition will come that this determinant or you can say ac minus b square must be zero if ac minus b square is not equal to zero then the only solution is zero zero
 so we need the condition the condition is ac minus b square is equal to zero
 so what are the cases for a b c you can have one one one one two 4 4 2 1 2 2 2
 so you can see here b square is 1 ac is 1 b square is 4 ac is 4 b square is 4 ac is 4 and
 so on
 so you have these cases 3 3 3 4 4 4 5 5 5 and 6 6 6.
 the total number of cases is eight eight cases are there
 so the required probability will be eight divided by now the total number of cases when we toss a die three times
 so it will become six cube that is two one six
 so if we simplify that is equal to one by twenty seven ah in this problem little bit of mathematical knowledge is also required but of course the system of linear equations ah at least two linear equations you are doing in your class ah eleven and 12
 so you should be able to solve ah in the next lecture i will solve several problems related to probability and there will be some problems which are taken from some of the question papers and i would request you to go through the chapter on permutations and combinations because some of the problems may use these things
 so it will be better if you ah go through the chapter on that
 so that ah the problems understanding will be much better
 so in the next ah lecture i will reword problems on ah various kind of applications of probability you