in the last class i have introduced ah basic concepts of probability and we have given certain definitions i call them the classical definition of probability the relative frequency definition of the probability and the exomatic definition of probability ah let me just repeat the last definition which i said eczematic definition of probability according to this probability is a function defined on the event space

so we say that probability of every event is always non negative probability of the full sample space is one and if i have a collection of pairwise disjoint events then the probability of their union is equal to some of the probabilities now which is called actually an axiom of additivity as a consequence of this we saw that the probability always lies between 0 and 1 probability is a monotone function that means if an event is more likely to occur then probability of that will be larger ah probability of ah a complementary event is equal to one minus the probability of the original event and probability of the impossible event is zero ah now you can consider this as a broad framework under which all the probabilities lie that means whether we calculate the probability using ah the classical definition of probability or if we calculate the probability using the relative frequency definition of probability it must satisfy the framework of the given by the axomatic definition ah i will continue with some of the rules ah which will follow from the asthmatic definition you may note that some of the proofs may be given in your book of class eleventh and twelfth but here i will be giving the proof specifically using the axiomatic definition that means the set theoretic construction of the probability that has been given will be used here

so the first rule which is the follow-up from the definition this is called the addition rule of probability the rule is as follows let a and b be any two events then of a union b is given by probability of a plus probability of b minus probability of a intersection b i will explain through the vein diagram suppose we consider this as the sample space and we have two events say a and b here suppose this is event a and this is event b

so the probability of a union b that is this entire thing is probability of this a and the probability of b minus the probability of a intersection b this is because a intersection b term has been added twice because in a a intersection b is included and a intersection b is also included in b that means when we are saying probability of a plus probability of b then we are adding this probability of a intersection b two times therefore we remove it once let me give a theoretical proof of this and you can see that using this set theoretic representation the proof is very simple

so let us see we can write the set a union b equal to

so we consider this part a

so this entire thing is this is a now if i add only this part the dotted part then i will get the entire a union b now if you look at this dotted part it is actually b from the b we are removing the portion a intersection b

so we can write it as a union b minus a intersection b

so let us see this set theoretic representation a union b is this entire thing this i am writing as union of two disjoint sets one set i take to be a itself which is this portion which is lined portion now the in the remaining part i have this dotted portion that is some part of b and what part of b here from the whole set b we remove this lined part which is actually a intersection b this is exactly equal to b minus a intersection b

so if i consider probability of a union b now i am saying it is equal to probability of these two union of these two disjoint sets therefore this will become probability of a plus probability of b minus a intersection b now we can look at this further what is the type of result i am having if you remember we

considered one result yesterday if i have f as a subset of e then i got probability of e minus f is equal to probability of e minus probability of f that means the probability of difference between two events is equal to the difference of the probabilities of the two events provided one of the events is a subset of another

so here we got this statement if f is a subset of e then we have probability of e minus f is equal to probability of e minus probability of f

so let us use this on this term in this term a intersection b is a subset of b therefore this becomes probability of a plus probability of b minus probability of a intersection b and if you look at read the statement completely now it is probability of a union b is equal to probability of a plus probability of b minus probability of a intersection b which is actually the addition rule

so you can see using this set theoretic representation and the axioms the proof of the statement is very easy it is extremely trivial

so ah this addition rule is used to calculate the probability of union of two events now the natural question arises in place of the union of two events if i have union of three events if i have union of ten events then what will be the extension of this

so we consider extension firstly two three and then i will show you that the extension to any number is also easy extension to three events say a b c

so we consider probability of a union b union c now in this i can consider a union b as a block

so we can write it as a union b plus probability of c minus probability of a union b intersection c

so what we have do done i have applied actually the addition rule which is given for two events on this by taking this as one event and this as another event

so it is probability of the first one plus the probability of the second one minus the probability of the first one intersection with the second one now on the first part here again i can apply the addition rule

so this i can write as probability of a plus probability of b minus probability of a intersection b then we have this probability of c let us look at this term here i am having a union b intersection c on this i apply the distributive property of the sets what is the distributive property of the sets this becomes a intersection c union b intersection c

so again i am having probability of one set union another set and again on this part i can apply the addition rule

so this becomes

so let me compile the terms here probability a plus probability of b and this third term plus probability of c minus probability of a intersection b now i have this term with a minus sign outside

so i will put it as a minus and i will put it in the parenthesis probability of a intersection c plus probability of b intersection c minus probability of a intersection c intersection b intersection c

so what i have done i have applied the addition rule on this particular term which gives me this

so if i collect all the terms i am getting probability a plus probability of b plus probability of c and now let us look at the terms which are involving two events

so you are having minus probability of a intersection b minus probability of a intersection c minus probability of b intersection c and then the last term will become plus because here you have a minus sign here

so this becomes plus probability of now you are having a intersection c and intersection with b intersection c

so if i open this parenthesis here it becomes actually a intersection b intersection c

so you have proved the formula for the addition rule for three events that means if i have three events a b and c then the probability of the union is given by firstly some of the probabilities taking one at a time then minus here i am considering probabilities of intersection of two of them

so intersection of a with b intersection of a with c intersection of b with c and then you are further doing plus here intersection three at a time now why that has happened we can actually try to understand it using some sort of ah wave diagram let us consider three events say a b c

so if i am considering the probability of the union i am looking at probability of a

so a is this full term then you are looking at the b and then you are looking at the c now from here this a intersection b which has taken twice has been removed then b intersection c which was taken twice has been removed and a intersection c which has been taken twice has also been removed but in the process this a intersection b intersection c has been removed one additional time because three times you added and three times you removed

so here that term is totally out

so this a intersections b intersection c should have been actually added

so that is justified here by the theoretical proof here

so if i have more than two events then also this addition rule is applicable in fact this gives you an idea that how a generalization will be there suppose i have four events if i have four events then the probability of the union will give me the formula probability of taking each of them

so that will be the sum then minus taking two at a time all the combinations of events taking two at a time

so four c two six such terms will be there and then plus three at a time

so four such terms will be there and then again with a minus all of them together now that gives rise to whether we can have a general addition rule the answer is yes ah now in mathematics proofs you have done something called the principle of mathematical induction ah i will show you that by using this principle of mathematical induction we can prove the general addition rule

so let us go for that general addition

so let a one a two and

so on a n b any events then probability of union of a i i is equal to one to n that is equal to summation probability of a i i is equal to one to n that is sums taking one at a time minus probability of a i intersection a j i less than j plus triple summation probability of a i intersection a j intersection a k i less than j less than k minus and

so on plus minus 1 to the power n plus 1 probability of intersection a i i is equal to 1 to n that means the last term will be taking all of them together and the sign will depend upon whether you have an odd number of events or even number of events

so if you have an odd number of events then the last term will become positive if you have an even number of events then the last term will become negative as you have seen when i considered odd number of events three events here then the last term was positive in the case of two this is even number of terms

so the last term is negative

so let us look at the proof of this i mentioned to you that i will use the principle of mathematical induction for this now let me call this relationship ah one to refer back we will prove the relation one using the principle of mathematical induction now let me just ah remind you what is the principle of mathematical induction in the principle of mathematical induction if we want to

prove a statement say p n for all n where n takes positive integral values then we should firstly prove that p one is true and then we assume that p k is true for n is equal to k and using that we prove that p k plus one is true let me repeat the steps firstly we show that for n is equal to one this is true and then we assume that for n is equal to k it is true and using that we prove for k plus one an alternative way or a secondary way of looking at it is that we prove it for one and we assume it up to k and then we use it to prove it for k plus one

 so let me write here the proof for this statement the which is given for the general addition rule

 so for n is equal to one what is the statement if i put n is equal to one here in the union i will have exactly one term that means it will become probability of a one and on the right hand side i will get exactly one term that is probability of a one

 so probability of a one is equal to probability of a one

 so the statement is trivially true

 so for n is equal to one the statement one becomes p of a one is equal to p of a one which always holds true

 so next we assume the statement one to be true for all n is equal to k

 so let us say for n is equal to k rather than saying for all n is equal to k

 so then we prove it for n is equal to k plus one

 so for k plus one what is the left hand term the left hand term becomes probability of union a i i is equal to one to k plus one

 so this we write as probability of union a i i is equal to one to k union a k plus one now here what i have done i have written it as union of two terms

 so this union from one to k i write as one event and another event is a k plus one now for the two we have already the addition rule

 so we apply the addition rule

 so i will get that is equal to probability of union a i i is equal to one to k plus probability of a k plus one minus probability of union a i a k plus one ok this is by addition rule for two events now if you look at the first term this is probability of union of k events and we have made the assumption that for n is equal to k the statement is true that means on this term we can directly apply the addition rule and write whatever term is there which is actually available through this formula only thing is in place of n we will write k here for all the terms we will put k

 so this then becomes

 so this is equal to sigma probability of a i i is equal to one to k minus double summation i less than j probability of a i less than a j and let me write here the upper term also just to denote that we have terms which are going ranging up to k only

 so i am putting one k here plus summation probability of a i intersection a j intersection a m i less than j less than m and this terms range up to k only and

 so on finally we have minus one to the power k plus one probability of intersection of a i i is equal to one to k now this term which i have written is basically the expansion of probability of union a i from one to k because we assumed the statement to be true for n is equal to k now the next term is probability of a k plus 1 which i write here as such let us look at the next term here it is intersection of a set taken with the union again i can apply the distributive property of the unions and intersections

 so this term becomes minus probability of union a i intersection a k plus 1 i is equal to 1 to k again you can see that it has become union of k terms and therefore the addition rules formula which has been assumed to be true for k events can be applied on this

so let me repeat the terms here here i have summation of probability of a i from one to k and here i have probability of a k plus one
 so this term i can add here
 so this first term becomes probability of a i i is equal to one to k plus one and then the remaining terms i will write as such i less than j ranging up to k probability of a i intersection a j plus probability of a i intersection a j intersection a m i less than j less than m up to k minus one to the power k plus one probability of intersection a i i is equal to one to k ah this term i already written combined with this now we are getting this term
 so i will put a parenthesis here let us put a square bracket this is probability of union of k events and i apply the addition rule for this
 so if i apply the addition rule for this it becomes summation probability of a i intersection a k plus 1 i is equal to 1 to k then minus double summation i less than j probability of a i intersection a k plus 1 intersection a j intersection a k plus 1 and this is ranging up to k and
 so on minus one to the power k plus one probability of intersection of a i intersection a k plus one i is equal to one to k ah let me read this term carefully if you are not been able to see it carefully this is probability of intersection a i intersection a k plus one because these are the sets that are available in the expansion that this is the term which i am expanding
 so the sets are of the type a i intersection a k plus one
 so the last term will include the intersection of all of them that is intersection a i intersection a k plus one from i is equal to one to k now we observe what are the terms here
 so let us look at this term remains as such let us come to this what are the terms here if i look at the terms this is probability of a one intersection a k plus one probability of a two intersection a k plus one probability of a three intersection a k plus one and
 so on up to probability of a k intersection a k plus one that means all the subscripts which are less than k plus one their intersection with a k plus one has been taken and there is a minus sign here you look at these terms here all the intersections are there for i less than j but this is only up to k that means you will have terms like a 1 intersection a two a one intersection a three a one intersection a k a two intersection a three a two intersection a k and
 so on up to a k minus one intersection a k plus one a k all these terms will be there
 so here since all the terms are up to k and now we have added one additional term that is a k plus 1 and all such terms are there
 so i can combine this with this term
 so that will give me
 so let me write the combined terms now this is probability of a i i is equal to one to k plus one minus double summation i less than j probability of a i intersection a j up to k plus one
 so that is the difference please note this difference here here we had up to k now we are having up to k plus one now let us look at the next one
 so here we are having intersection of three events all the terms taken up to k that is i may have terms like a 1 intersection a 2 intersection a 3 a 1 intersection a 2 intersection a 4 a 1 intersection a 2 intersection a k similarly a two intersection a three intersection a k and
 so on up to finally i will get the terms a k minus two intersection ak minus one intersection a k
 so all such terms will be there taken three at a time where the subscripts run up to k now let us look at and this is with a positive sign now let us look at this term here here it is a i intersection aj intersection a k plus one because

a k plus one is coming at two places

so i and j subscripts are for one to k and then you are taking intersection with k plus one that means i will get terms like a one intersection a two intersection a k plus one a one intersection a three intersection a k plus one and

so on a k minus one a k intersection a k plus one that means in this all the terms are coming such that this will become up to k plus one that is three at a time

so i can write this here plus summation probability of a i intersection a j intersection a k

so let me put here a m i less than j less than m up to k plus one

so what we are observing that these terms which were up to k are getting extended up to k plus one now all these terms

so similar thing will happen with all the terms you can show and let us look at the last terms now

so the last term here is intersection of all the a i's for one to k and then intersection with a k plus one

so basically it becomes intersection of all the terms that is all a i's for i is equal to one to k plus one let us look at the sign of this this is minus one to the power k plus one and there is an additional minus outside

so this will again be combined with this to give us plus minus one to the power k plus two probability of intersection a i i is equal to one to k plus two

so if i look at what i have proved we have actually written probability of union a i i is equal to one to k plus one equal to sum of all the probabilities taking one at a time minus sum of all the probabilities taking two at a time plus sum of all the probabilities taking three at a time minus and

so on and finally probability of intersection of all the events which is exactly the statement which i wrote for this n if i replace n by k plus 1 here that is the statement i will be getting here

so this shows that statement true statement one is true for n is equal to k plus one hence by the principle of mathematical induction the general addition rule holds for all n where n is a positive integer ah

so these rules are used for

so i have actually given certain consequences from the ah axiomatic definition and the first consequence or you can say first important consequences that we can calculate the probability of union of certain number of events

so this type of ah formula is extremely useful i will show one example ah just to show how we can use it for calculation of ah probabilities where certain complex events may be there

so actually i will apply the classical definition for calculation of the basic probabilities and then we will apply this addition rule

so let me take one such example suppose six cards are drawn one by one with replacement from a well shuffled pack of 52 cards ok

so let me repeat the language here

so the terminology with replacement means that we draw a card we note down what is the card and we put it back in the deck and again we take another card again note down what is the card and again put it back in the pack of cards

so six times this experiment is repeated we want to find out the probability that in this set of six cards each of the four suits that is heart spade club and diamond appear

so each of the four suits heart is paid club and diamond appear in this set of six cards that means no suit is unrepresented that means i do not have the situation where only heart is there or only heart is not there or speed is there our speed is not there or two of them are there or two of them are not there

whatever be the sex set of six such cards all four will be there that means some may be more than one also because total six are there

so maybe you have ah two hearts two spades one club and one diamond etcetera

so what is the probability of this ah

so you can go and directly enumerate also i will show you that if we use this addition rule the calculation of this probability becomes quite simple

so i am solving this problem using the general addition rule

so let a be the event that the set of six cards contains at least one card of each suit

so then what is a complement a complement will mean that at least one suit does not in the set of six cards the purpose of doing like this is that i will show you that first of all we are using a set theoretic representation of the events ah if we do not know the set theory representation then we can start counting straight away as i mentioned you calculate the possibility three hearts one spade one club one diamond three speeds one heart one club one diamond three clubs one hard one spade one diamond and

so on then two hearts two spade one club one diamond like that

so you can look at all the possibilities calculate the probability of each of them and then add

so that is the straight forward fashion but then you do not need to really define many events what i am trying to do here is to use the set theoretic notation and apply the addition rule and you will see that the answer becomes very nicely calculated here

so i am taking a compliment as the event that at least once you does not appear in the set of six cards

so let us consider then event b one by saying that say heart do not appear then you can write b two as saying say spades do not appear b three as the event say clubs do not appear and b4 say saying diamonds do not appear then we can write a compliment as union of b i i is equal to one to four because what is the meaning of union occurrence of at least one of them

so here i said a complement is at least one suit does not appear

so here since b1 b2 b3 b4 denote that one of them does not appear

so union will mean at least one of them does not appear

so this is an exact representation of a complement

so if i apply the general addition rule then probability of a complement will become probability of the union and for the union of four events i apply the general addition rule now for application of this i will need to calculate probabilities of b 1 b 2 b 3 b 4 probability of b 1 intersection b 2 b 1 intersection b 3 and

so on probability of intersection taking three at a time and probability of intersection of all of them

so let us look at this by general addition rule probability of a complement is equal to probability of union b i that is equal to sigma probability of b i i is equal to one to four minus i less than j up to four probability of b i intersection b j plus probability of b i intersection b j intersection b k i less than j less than k up to four minus probability of intersection of all of them

so this is the addition rule for four events for the union of four events i need to calculate the terms in each of these summations here

so let us start with the first one i consider what is the probability of b one ok

so let me keep this here to show you that what are the terms that i am actually enumerating

so first of all let us look at what is the probability of b one now b one is

the event that hearts do not appear b one is the event hearts do not appear what does it mean if i am considering six times the cards have been drawn in the first one it is not a heart second one is not a heart and
 so on up to sixth one it is not a heart
 so if i consider the first one is not a heart then what does it mean out of total number of cards you have thirteen cards of heart total number of cards are fifty two
 so you are saying that in one draw you are drawing any card which is other than the heart that means the card has been drawn out of the remaining 39 cards
 so the probability of drawing a card which is not a heart it will become 39 by 52 that is 3 by 4.

 so in one ra the probability that it is not a heart that becomes three by four now this thing is repeated six times because you are putting back the card
 so next time also the calculation of the probability will be same because next time again you have 52 cards out of which there are 13 cards which are not hard
 so again it will become 3 by 4 and you will be actually repeating this six times
 so basically you are getting three by four to the power six now if i am considering b2 then b two is if spades do not appear the calculation of the probability for b two will be same as what i gave the argument for that hearts do not appear because if spades also there are thirteen cards
 so if in a draw there is no spade then the probability will be three by four
 so we can actually give the statement that probability of b i is equal to three by four to the power six for i is equal to one two three and four
 so these are the terms which are actually included in this probability of b i because here i need probability of b one probability of b two probability of b three and probability of b four
 so all the terms are calculated they are all same as three by four to the power six
 so in the final calculation i will be putting the value 4 into 3 by 4 to the power 6 let us look at the next term now this includes probability of b 1 intersection b 2 b1 intersection b3 b1 intersection b4 b2 intersection b3 and
 so on out of 4 you are taking 2 at a time
 so the number of terms will be 6 that is four c two that is four combination two
 so there will be six terms which will involve probability of b i intersection b j let us look at the calculation of this suppose i write probability of b one intersection b two
 so b one intersection b two means that hearts and spades do not appear now in the total collection of 52 cards there are 26 cards which are hearts and spades
 so you are saying they do not appear
 so in one draw the probability will be half that it is not a heart and not a spade then second draw since your place the card
 so the deck is again complete the probability remains the same
 so six times you are doing
 so it actually again becomes using the previous argument half to the power six and actually you can write probability of b i intersection b j that is equal to half to the power six where i is less than j
 so there are total such four c two that is equal to six terms are there then in the next one there are three terms
 so you are saying three of the types do not appear that means i can say hearts spades and clubs do not appear basically it means you are saying only diamonds appear

so if only diamond appear the probability will be one by four and you are doing it six times

so in general i can say probability of b i intersection b j intersection b k will be one by four to the power six for i less than j less than k

so there are total such four terms are there you have four terms here six terms here and four terms here now let us look at the last term the last term is intersection of all the four events but what are the events the events are hearts do not appear spades do not appear clubs do not appear and diamonds do not appear

so roughly you are saying that nothing appears which is not possible because when you draw a card it will be one of these

so the probability of intersection b i becomes zero

so probability of intersection b i i is equal to one to four is equal to zero now in this formula i have evaluated all the terms

so if i substitute here i will get

so probability of a compliment becomes 4 times 3 by 4 to the power 6 minus 6 times half to the power six plus four times one by four to the power six

so one can simplify this and we get the term equal to three one seven by five one two which is approximately point six two and you can calculate probability of a that is one minus probability of a that is equal to one ninety five divided by five one two that is approximately point three eight ah apart from doing this calculation actually i have shown you an application of the general addition rule but apart from that let us also appreciate the numerical value that i am writing here

so when we are considering drawing of six cards one by one with replacement there is a sixty two percent chance that means more than sixty percent chance that at least one of the suits is not represented and similarly here if i am drawing six cards there is less than forty percent chance that each of the suits will be represented at least once

so ah in fact what is the general feeling that if i have actually ah four types of suits and we are drying six times

so naturally there is a feeling that there is will be a high probability that each of them will be appearing at least once but as you can see the value is less than 0.

4 that is less than 40 percent chance that each of them will be represented

so ah actually one of the basic purposes of calculation of the numerical value for the probability is to have a feeling for the how much chance we are having

so like we give lose statement there is a 90 percent chance that it will rain ah tomorrow or it will be very cold tomorrow this kind of statements we give

so this 90 percent term that we are saying

so that is something like denoting a probability

so the actual calculation of the probabilities using the rules tells you that how much belief or how much trust you can give in such statements ah

so i have shown you one simple application ah now i will give you one or two ah new definitions here let us consider say suppose i consider a die is rolled ok and let us consider it to be a fair die ok i consider say an event say i say what is event ah a suppose i say one occurs that is the upper face is one then what is the probability of a it is one by six i define another event b and i say odd number occurs then what is the probability of b it is half because odd number means one three five now i give another statement what is the probability that one occurs given that an odd number occurs now you see i have modified my statement i already know that an odd number has occurred

so here my sample space has become much less it is only three terms and assuming that fair if i calculate the probability of one it will become one by

three
 so what is the probability of a given that b occurs or b has occurred that is equal to one by three this is the concept of conditioning
 so i call it conditional probability you can see here the probability of event a is actually one by six that is what is the probability in the tossing of a dice it is one by six but if i am considering that an odd number has occurred then what is the probability of one then it becomes one by three that means if there is an additional information in a random experiment then the probabilities get modified this concept is ah given by conditional probability
 so in the next lecture i will introduce ah conditional probability and based on conditional probability there will be certain rules and certain theorems which i will explain and then we will go to solve certain problems on this okay you