so friends in the last class i have introduced the concept of random variables we have seen that the random variable can take values which are finite in nature like 1 2 3 etc or they can be countably infinite like we can say 1 2 3 and
so on or they can take values over an interval for example height weights price etc
so we accordingly distinguish them as discrete or continuous random variables uh i have elaborated upon discrete random variables and how to describe the probability distribution of that based on the probability distribution we look at the concept of the mean or the expectation or the expected value of a random variable and the variability variability is measured in terms of variance of a random variable which i define let me further explore it little bit
so let us consider we can consider alternative formula for evaluation of the variance
so let us consider what formula i wrote
so i wrote variance of x is equal to expectation of x minus mu square where mu itself denotes the mean of x are the expectation of x let us expand it
so this is equal to expectation of x square minus 2 mu times expectation x plus mu square now here i will try to use some properties of the expectation
so let me call it 1 to simplify one we will use some properties of expectation the first property is that expectation of a constant is constant okay this is simple to see because expectation of c you can say the value multiplied by the probability
so here whatever be the value of random variable x it is actually
so p 1 p 2 and
so on p n say suppose the random variable is taking these probabilities
so it becomes c p 1 plus p 2 plus p n that is equal to c into 1 that is equal to c
so expectation of a constant is the same constant similarly if i consider random variable x and i consider a linear function of that then it is equal to a times expectation of x plus b
so this is also very easy to see if i write expectation of ax plus b then it is equal to a x 1 plus b into p 1 plus a x 2 plus b into p 2 plus and
so on a x n plus b into p n
so this you can expand a times x 1 p 1 plus x 2 p 2 and
so on plus x n p n that is this term plus b times p 1 plus p2 plus pn
so this is nothing but expectation of x
so you get a times expectation of x plus v times p1 plus p2 plus pn is 1.

so this means that expectation is a linear function okay
so if we use this property in this expression then we will get
so using these properties in 1 we get variance of x is equal to
so this is expectation of x square
so this will go inside expectation of x square minus 2 mu times expectation of x plus expectation of mu square which is mu square because mu square is a constant
so this expectation of x minus mu square has now become this value
so we can write it as expectation of x square minus 2 mu and what is expectation x that is mu
so it becomes 2 mu square plus nu square that is equal to expectation of x square minus mu square
so this is an alternative formula for calculation of the variance x that means we can say variance of x is equal to expectation of x square minus expectation of x whole square ah this formula is sometimes very useful in calculation of the

expectations because ah given a random variable it is easier to calculate expectation x and expectation of x square because this is direct one whereas in the last one you have seen that expectation of x minus mu square if you calculate the number of terms to evaluate you have to consider lot of differences and their squares etc

so let me use this and calculate variance in one of the earlier problems

so let us consider that score of the card problem

so let me show that example once again and we are having the value here the score of the card problem had the distribution probability x equal to y is equal to one by 13 for i is equal to 2 to 10 probability x equal to 15 is 3 by 13 and probability x equal to 18 is 1 by 13.

we had already calculated expectation of x that was equal to 9.

now for the same one if i calculate what is expectation of x square

so expectation of x square will become sigma i square 1 by 13 i is equal to 2 to 10 plus 15 square into 3 by 13 plus 18 square into 1 by 13 ah you can actually try to compare it with the expression for expectation x

so whatever value x is there in expectation x square we are putting x square and p i's are the same

so this is easy way of looking at the calculations now in order to evaluate this we apply the formula for the sum of squares of first n integers

so first n integers you have the formula n into n plus 1 into 2 n plus 1 by 6 from that sum the first term is removed because here it is from 2 to 10

so i is equal to 1 is not there

so this becomes then you can actually look at the calculation 10 into 11 into 21 by 6 minus 1 and 1 by 13 let me keep outside plus 225 by 13 into 3 by 13 plus 324 by 13.

so one can easily simplify all these terms it becomes 1 3 8 3 by 13.

so now variance of x is equal to expectation of x square minus expectation of x whole square that is 1 3 8 3 by 13 and expectation x was nine here

so we look at this nine square

so after certain simplification it is equal to three hundred thirty by thirteen ah now you can see that this variance term it is actually giving you the squared deviation from the mean

so one reasonable measure to bring it to unit of measurement because now it is squared units of measurement

so if we take square root of that that gives you the variability in terms of the same units of measurement for example if you are looking at centimeters you are looking at kilograms or if you are looking at liters then the units must be same for description

so we consider the standard deviation

so we define a standard deviation of a random variable x is defined

so we write s d of x is defined to be square root of variance of x

so let me look at some more problems here two distinct fair dice are rolled once let x denote the absolute difference of the numbers shown on the upper faces of the two dice

so we want to find out the distribution of x and say what is variance of x when the two dies are rolled the number of possibilities are 36 you are having the sample space 1 1 1 2 2 1 2 2 2 6 3 1 3 2 3 6 and

so on 6 1 6 2 6 6.

so if we want to look at the definition of the random variable x it is the difference if we have 1 1 the difference is 0 if it is 2 2 the difference is 0

if it is 1 2 then the difference is minus 1 and absolute difference will become 1 if it is 3 6 then the difference is minus 3 and the absolute difference becomes 3 and
 so on
 so the possible values the possible values of x that is absolute difference they are 0 1 2 3 4 and 5.

 so you can say the absolute difference is 0 for 1 1 2 2 3 3 4 4 5 5 and 6 6 there are total 6 cases
 so what is the probability that x equal to 0 it becomes 6 by 36 that is 1 by six now the absolute difference one what are the possible cases you can have one two and then of course two one then you can have two three three two three four four three four five five four five six and six five total number of cases is n cases
 so what is the probability of x equal to one that becomes 10 by 36 you can also simplify it by writing 5 by 18.
 now let us look at the absolute difference two one three three 1 2 4 4 2 3 5 5 3 4 6 6 4.

 so you have total eight cases
 so probability that x equal to two that will become eight by thirty six that is equal to two by nine now let us look at the absolute difference three one four four one two five five two three six six three
 so you have total six
 so the probability that x equal to 3 that will become 6 by 36 that is 1 by 6.
 now the absolute difference is 4 for 1 5 5 1 2 6 6 2 there are total 4 cases
 so the probability that x equal to 4 that will become 4 by 36 that is 1 by 9.
 similarly let us look at the absolute difference five one six and six one that is total two cases only
 so the probability of x equal to five is equal to two by thirty six that is equal to one by eighteen you can cross check that it is a valid probability distribution the total probability is six plus ten sixteen plus eight twenty four plus six thirty plus 434 plus 236 by 36 that is 1.

 so this is the probability distribution of x we can write it like this the probability distribution of x is p 0 is 1 by 6 p 1 is equal to 5 by 18 p 2 is equal to 2 by 9 p 3 is equal to 1 by 6 p 4 is equal to 1 by 9 and p 5 is equal to 1 by 18 this is a complete probability distribution of the random variable which is described as the absolute difference on the upper two faces observed when two fair dice are rolled suppose i want to calculate the expectation here
 so that is becoming 0 into 1 by 6 plus 1 into 5 by 18 plus 2 into 2 by 9 plus 3 into 1 by 6 plus 4 into 1 by 9 plus 5 into 1 by 18
 so one can simplify this it is 35 by 18 you can see that actually it is slightly less than two if we calculate expectation of x square then it is zero square into one by six plus one square
 so that second value plus 2 square into 2 by 9 plus 3 square into 1 by 6 plus 4 square into 1 by 9 plus 5 square into 1 by 18 once again you can simplify it is equal to 35 by 6
 so the variance of x is equal to expectation of x square minus expectation of x whole square that is 35 by 6 minus 35 by 18 square
 so this can be simplified it is equal to 665 divided by 3 to 4 which is little more than two let me give one more example of a discrete distribution and on contains n distinct marbles marbles are drawn successively with replacement from the on at random until a mr bill is repeated

so for example we can put some tags on those marbles

so one marble is drawn and its tag is noted the marble is put back in the arm and then another marble is drawn now it can be the previous one or it can be another one if it is the previous one we stop otherwise we continue

so then again we put it back and draw another one

so again it can be one of the two previous one or it can be a new one like that we continue till we get something which is drawn before

so let us consider x with the number let x denote the number of draws needed to complete the experiment

so we want to find out the distribution of x

so x can take values two three and

so on up to n plus one since there are total n marbles which are distinct therefore certainly at the n plus first draw you will have repetition after that there is no need because if in all the trials you get a distinct one then certainly on the n plus one h trial you will get the one which has been drawn before

so what is the probability that x is equal to two that means on the first draw whatever marble was drawn it is drawn again

so if we draw one and that tag is noted then it is one of those n

so in the second one if we are drawing that the probability of that will be one by one

so we can write it that is the marble drawn on the first draw is drawn again what is the probability x equal to 3 that means in the second trial we got a distinct one now in the first trial you draw one and we remember the tag in the second one that is not drawn that means whatever is drawn is from the remaining n minus 1

so the probability of that is n minus 1 by n now in 1 and 2 draws two distinct marbles are there and in the third one we draw one of that

so the probability of that will be 2 by n

so we can say that is the marble on second is distinct from the first and third is any of the first two let me repeat this argument here x is equal to means that whatever is drawn on the first is again drawn on the second

so if we are considering sumdra on the first one then that is fixed out of n one is fixed and we are trying it again

so the probability of that is 1 by n in the second case if x equal to 3 that means in the second draw we get something distinct from the first one

so the probability of that will be n minus 1 by n and then we get something which is from the first two

so that is two by n

so like that we can continue let me write first few terms what is the probability x equal to four

so that will be first one is distinct ah second one is distinct from the first one the third one is distinct from the first two

so n minus two by n and then it is one of the first three at the i-th one n minus 1 by n n minus 2 by n and

so on n minus i plus 2 by n and i minus 1 by n and

so on what is the probability x equal to n plus 1 that is n minus 1 by n n minus 2 by n and

so on up to 1 by n finally n by n we can verify that sigma probability x is equal to i i is equal to 2 to n plus 1 is 1

so actually for checking that you have to use the backward summation start from the end and use term by term sum now i have discussed uh in detail one particular kind of random variables they are called discrete random variables

so the discrete random variables are the ones which take finite or countably

infinite number of values in the probability theory there are certain specific random variables which are used commonly and they are nice to describe various physical situations one of them is the famous binomial distribution

so in your syllabus also in the class 12 you are actually studying the binomial distribution

so let me describe that and this binomial distribution is named after jacob bernoulli the swiss mathematician and his published book ours conjecturally is one of the most you can say popular books and it is one of the fundamental books of the probability theory here he described some experiments which are dichotomous in nature that means in which the number of possible outcomes are two

so let us consider for example somebody falls sick and he goes to a doctor and gets the medicine then the outcome can be that he gets treated or he doesn't get treated from that medicine that means you are looking at two outcomes getting treated or not getting treated somebody appears in a competitive exam

so he may get some marks but we may be interested in knowing whether he qualifies the exam or he does not qualify the exam a shooter hits at a target

so we are interested in knowing whether the outcome is a success that means he successfully hits the target or he misses the target

so likewise there are large number of real life situations where the experiment may be complex but we are interested only to look at two outcomes because that may be of ah some interest for recording purpose

so such experiments are called bernoullian trials or bernoullian experiments in many in real life experiments one is interested in only two possible outcomes for example hitting a target

so success or failure two outcomes are there treating a patient cured not cured appearing in exam

so you can say qualifying or not qualifying

so in such experiments we specify one outcome as success and another outcome as failure

so let me write s for success and f for failure okay and these are called bernoullian trials

so let us make an assumption suppose n bernoullian trials are performed under identical conditions and independently of each other and suppose the probability of success is same in all trials say p and

so probability of failure is also the same in all trials say one minus p that is q we can write ok

so here this p and q will be numbers between zero and one okay let x denote the number of successes in n trials then x is a discrete random variable what are the possible values of x x is a discrete random variable and it can take values 0 1 2 and

so on up to n you are conducting the trial n times

so you can have all the failures one success two success are all the success

so the possible values of x can be 0 1 2 up to n what is the probability of say x is equal to k now you are having let me just draw it here these are the trials ok 1 2 3 and

so on

so now you are saying that out of n trials k of them are success the trials are conducted independently in one trial the probability of success is p

so if you consider k success the probability becomes p into p up to k times that becomes p to the power k but it also means that remaining n minus k trials they result in failures now the probability of one failure is 1 minus p or q and again the trials are independent

so it becomes q into q n minus k times

so it becomes q to the power n minus k we are calculating what is the probability that out of n bernoullian trials they are k success and n minus k failures consequently

so now there are total n trials now this k values which are resulting as success can be any of these k

so the number of ways of choosing that will be n c k

so k is equal to 0 1 2 n

so you are describing let me call it p k

so p 0 p 1 up to p n you have allocated the probabilities corresponding to all the possible values of the random variable x this is called binomial distribution also it is called bernoulli distribution

so binomial distribution name is evident because you are using the binomial coefficients here first of all i will try to demonstrate that it is a valid assignment of the probabilities we first show that this is a valid probability distribution that is sigma p k k equal to 0 to n this must be 1

so let us look at this this is nc k p to the power k q to the power n minus k k equal to 0 to n

so that is equal to if i write all the terms it is n c 0 q to the power n plus n c 1 p q to the power n minus 1 plus n c 2 p square q to the power n minus 2 and

so on plus n c n p to the power n if you remember your binomial theorem then from the binomial theorem this is nothing but the expansion of q plus p to the power n now q is 1 minus p

so q plus p is 1

so it is 1 to the power n that is equal to 1

so sum of all the probabilities is 1 and all the probabilities are non negative therefore this is a valid assignment of the probabilities ah i already introduced the concept of expected value or the mean and the variance

so let us see for the case of binomial distribution what are these values okay

so expectation x that is the mu that is equal to by the definition it is the value multiplied by the probability that is k into p k k equal to 0 to n now this is n c k p to the power k q to the power n minus k k equal to 0 to n you can see that in the summation here the first term is corresponding to k equal to zero

so when you put k equal to zero this term is actually zero

so we can actually say it is equal to k equal to one to n k into n c k p to the power k q to the power n minus k in order to evaluate this we need to expand this permutation combination notation

so that is equal to summation k equal to 1 to n k n factorial divided by k factorial n minus k factorial p to the power k q to the power n minus k when we use this notation we assume that this factorial notation

so factorial 1 is 1 factorial 2 is 1 into 2 and

so on and we also make assumption that k for 0

so 0 factorial we take trivially to be 1 as a convention

so that with this notation remains valid

so we can write it as n factorial divided by k minus 1 factorial and this term i write as n minus 1 minus k minus 1

so this numerator i write as n factorial p to the power k q to the power n minus k k equal to 1 to n

so here i will make some change in the notation let us put k minus 1 is equal to m then this becomes m is equal to 0 to n minus 1 this n i can take out

so you will get n minus 1 factorial p to the power k minus 1 q to the power n minus 1 minus m divided by m factorial n minus 1 minus m factorial this p i have taken out

so what i did here let us compare these two terms in this term i wrote n factorial as n into n minus 1 factorial and this 1 n i have taken outside similarly this p to the power k minus 1 i have written as p to the power k minus 1 and 1 p i take outside now this term is q to the power n minus 1 minus k minus 1

so k minus 1 i am writing as m

so this becomes n minus 1 minus m

so this term then becomes np sigma m is equal to 0 to n minus 1 n minus 1 choose m p to the power k minus that is becoming p to the power m now q to the power n minus 1 minus m

so this becomes n p and now this term if you see it is nothing but the expansion of q plus p to the power n minus 1 now which is 1

so it is n p

so what we have proved that the mean of the binomial distribution is n p let us try to physically interpret it in one throw of a coin or you can say in one trial of the bernoullian experiment the probability of success is p x denotes the number of successes

so what is the average number of success our expected number of success that is n into p which is natural to thing because if p is the proportion then np denotes the out of total how many are coming ah let us look at variance of this distribution now for variance we have used the formula expectation of x square minus expectation of x whole square now to calculate expectation of x square we further resolve it by writing it as the reason is that if you look at the expression for expectation x the simplification is done by cancelling out one of the factorial terms

so if i have a square term then that cannot be cancelled

so that's why it is better to consider it in this particular fashion

so that is equal to expectation of x into x minus 1 plus expectation of x expectation x has already been calculated it is np

so now i calculate expectation of x into x minus 1 k into k minus 1 n c k p to the power k q to the power n minus k for k equal to 0 to n you note here that corresponding to k equal to 0 and corresponding to k is equal to 1 this term becomes 0.

so we can write this summation from k equal to 2 to n k into k minus 1 n choose k p to the power k q to the power n minus k

so that is equal to k into k minus 1 k equal to 2 to n n factorial divided by k factorial n minus k factorial p to the power k q to the power n minus k in this one i cancel out this k into k minus 1 from the first two terms of this k factorial that is k into k minus 1 into k minus 2 factorial

so i get the term n factorial divided by k minus 2 factorial n minus k factorial p to the power k into q to the power n minus k in order to resolve this i consider the breakup like this n factorial i write as n into n minus 1 and here i will get n minus 2 factorial divided by k minus 2 factorial and this term can be written as n minus 2 minus k minus 2 factorial this i write as p to the power k minus 2 and p square i write outside into q to the power n minus 2 minus k minus 2 k equal to 2 to n once again i define k minus 2 is equal to say n then this becomes n into n minus 1 p square sigma m is equal to 0 to n minus 2 n minus 2 factorial divided by m factorial n minus 2 minus m factorial p to the power m q to the power n minus 2 minus m

so this is becoming n into n minus 1 into p square and this is nothing but the expansion of q plus p to the power n minus 2 the binomial expansion

so this q plus p is 1

so this becomes n into n minus 1 p square

so let us go back to the expression here expectation x square was expectation x into x minus 1 plus expectation x

so we get expectation of x square as n into n minus 1 p square plus n p

so variance of x that is equal to expectation of x a square minus expectation of x whole square that is equal to n into n minus 1 p square plus n p minus n square p square

so we can do little bit of simplification here the first term here is n square p square and here it is minus n square p square

so that cancels out

so you get n p minus np square that is equal to np into 1 minus p or you can say n p q

so interestingly now we have calculated expectation of x as np and the variance of x as npq where x has a binomial distribution something you can note here q is a number between 0 and 1.

so npq will always be less than np

so you can actually make a statement here as q is between 0 and 1 npq is less than np

so in a binomial distribution the mean is always more than the variance we can also write down the expression for the standard deviation that is the square root of n p q now we can also look at the special case when p is equal to half that means when the success and failure probabilities are same then what are the expressions then the success and failure probabilities are the same that is p is equal to q is equal to half then the distribution takes a very simple form that is probability x is equal to k that is equal to nc k half to the power n for k equal to 0 1 to n in this case if you look at the plotting here if i consider probability x is equal to say n minus k then that will be n c n minus k half to the power n that is n c k half to the power n that is probability x equal to k

so when the success and failure probabilities are same then the distribution allocates equal probabilities at the points which are symmetric that means if i consider 0 and n then the probability will be same if i consider 1 and n minus 1 then the probability will be same

so i let me just write it

so for example p 0 is equal to p n that will become simply 1 by 2 to the power n if i consider p 1 and p n minus 1 then that will become n divided by 2 to the power n similarly you can consider other values

so this is one special case when the success and failure probabilities are same and in this case expectation of x becomes n by 2 and variance of x becomes n by 4 and the standard deviation becomes square root n by 2.

in the next class i will consider various applications of the discrete distributions ah

so that means including the evaluation of expectations variance how to calculate various probabilities related to binomial distribution and some other ah discrete distribution

so i will spend sufficient time about the problems on these distributions you