

hello

so we come to the last and concluding lecture in the series on differential equations

so today's lecture we'll be looking at some odds and ends or maybe is a head if you like

so the issues that we discussed today will be as follows some uses of differential inequalities but this will be very similar to the way we worked with linear differential equations and then will converge towards the uniqueness theorem but we shall not exactly reach there we shall not state the uniqueness theorem the pendulum equation again we started this lecture series by deriving the pendulum equation that that was the first differential equation that we derived and we shall also end by looking at the pendulum equation a little more closely and then a few concluding remarks and

so let's begin how to use differential inequalities now let us look at this differential equation 4.

$1 \frac{dy}{dx} = y^2 + x$ with the initial conditions $y(0) = 0$
you see this differential equation 4.

1 is a Bernoulli equation it's a Bernoulli equation and

so you want to divide by y^2 but unfortunately you cannot why because $y(0) = 0$ and division by 0 is not allowed

so how would you solve 4.

1 well you see that the $y = 0$ solution is already a solution take the constant solution $y = 0$ it satisfies the differential equation the derivative of $y = 0$ is 0 and the right hand side is also 0.

so plug in the constant solution $y = 0$ in 4.

1 and you see at once that it's a solution of the differential equation it also satisfies the initial condition $y(0) = 0$.

but have we completely solved 4.

1 how do we know that there are no other solutions of 4.

1 satisfying $y(0) = 0$ maybe there are others besides the $y = 0$ solution which will satisfy 4.

1 how do we rule out that possibility we need a general theorem are you a general uniqueness theorem that guarantees that the zero solution is the only solution of 4.

1 and that there are no other solutions

so you see the need for a general uniqueness theorem already in this very simple situation

so let us go back to the very first lecture where we discussed differential inequalities where we knocked off some terms of the differential equation and got a differential inequality when we were discussing escape to infinity in finite time well

so let's look at the differential equation 4.

1 $\frac{dy}{dx} = xy + y^2$ the y^2 term is always positive

so let us look at the differential equation $\frac{dy}{dx} = xy + y^2$ the y^2 term is always positive

so let's see

so we can write $\frac{dy}{dx} - xy \geq 0$ I have knocked off the y^2 term from the differential equation four point one for the four point one I have knocked off the y^2 term

so from the differential equation I got the differential inequality 4.

2 $\frac{dy}{dx} - xy \geq 0$

now how do we proceed now suppose if in 4.

2 you're an equality instead of an inequality then 4.

2 would be a linear differential equation right

so how would you proceed in that case you would multiply the 4.

2 by e to the power minus x squared by 2 right but do the same thing here never mind it's an inequality but e to the power minus x squared by 2 is always positive

so i can jolly well multiply 4.

2 by e to the power minus x squared by 2 and the left hand side of 4.

2 becomes an exact derivative namely $\frac{d}{dx}$ of y of x into e to the power minus x squared by 2 greater than or equal to 0 which is 4.

3 in the displayed slide well what we do now we have to integrate 4.

3 we have to integrate 4.

3

so let's proceed you got the following situation let's see you got a function ϕ of x which is always non-negative on an interval a b then we can say $\int_a^b \phi(x) dx$ is surely non-negative you will agree with me on that why because what is the integral the integral is the area under the graph the area under the graph y equal to ϕ of x between x equal to a x equal to b and y equal to 0 but what does this inequality tell you the inequality tells you that the ϕ of x the graph of ϕ of x lies above the x axis the graph lies above the x axis then the area under the graph is going to be always positive and that's all that you need to say to prove this

so in particular if you have two functions $f(x)$ and $g(x)$ and if $f(x)$ is greater than or equal to $g(x)$ on the interval a b then $\int_a^b f(x) dx$ will be greater than or equal to $\int_a^b g(x) dx$ how do you how do you get this you get this from the previous one simply take ϕ equal to f minus g simply take ϕ equal to f minus g in the previous one and you get this one okay

so let's apply this

so let us go back to the slides you see you got 4.

3

so i say that integral on the left hand side will be greater than or equal to integral on the right hand side in other words the integral of the derivative from 0 to x i am going to integrate both sides of 4.

3 from 0 to x okay

so i am going to integrate both sides from 0 to x when you integrate a derivative you use the fundamental theorem of calculus

so you got $\frac{d}{dx}$ of y of x e to the power minus x squared by 2 greater than or equal to 0

so $\int_0^x \frac{d}{dt}$ of y of t e to the power minus t squared by two dt greater than or equal to zero of course in the definite integral the variable is a dummy variable

so lets use the fundamental theorem of calculus y of x e to the power minus x squared by 2 minus y of 0 e to the power minus 0 square by 2 greater than or equal to 0 this is 1 and y of 0 remember was 0 therefore we get y of x is greater than or equal to e to the power x squared by 2 into 0 which is 0.

so we have concluded that the solution is non-negative we have concluded that y of x is non-negative if x is greater than or equal to 0.

now we shall show that y of x is actually equal to zero let's proceed now we know that y of zero is zero and y is continuous and therefore if the value of the function at the origin is 0 then the value of the function must be less than 1 in a certain piece around 0 namely 0 to c

so on the interval 0 to c we know that y of x must be less than one if y of x is less than one and y of x is already non-negative y of x squared must be less than or equal to y of x

so let us go back to the differential equation what is the differential equation say it says $\frac{dy}{dx} = x + y^2$ but now we have seen that y^2 is going to be less than y

so let's use this inequality now

so since y^2 is less than y from 4.

1 we get the differential inequality $\frac{dy}{dx} \leq x + 1$ on the piece 0 to c

so again do the same thing again imitate as if you are solving a differential equation proceed as though you are going to solve a linear differential equation namely you multiply the differential equation by $e^{-\int p(x) dx}$ what is $p(x)$ in this case minus $x + 1$ and then you will conclude exactly as we proceeded just now y of x is less than or equal to zero and

so combining the two we see that y of x must be identically zero on this piece zero to c but now we want to show that not only is y of x identically zero on the piece zero to c it must be 0 everywhere you can actually show that is going to be 0 everywhere proceed by contradiction suppose it is 0 up to a certain interval after which it becomes positive then try to arrive at a contradiction be slightly careful if you are working this proof out ok

so now let me give you couple of exercises try the same idea on these two differential equations that you see in the slide the same idea of knocking off some terms getting an inequality and proceeding

so i worked out one example in detail and i'm asking you to try out two others

so the two differential equations are $\frac{dy}{dx} = y + \sqrt{1+x^3}$ the other one is $\frac{dy}{dx} = y + \sin^2 x$ for both these differential equations y of 0 is 0 .

and

so by inspection you see that 0 is a solution what you need to establish is that it is the only solution

so proceed as yeah i have indicated in the last exercise

so what's the point of this exercise isn't there a simpler and more direct approach to arrive at this conclusion each time you encounter a situation where you have to divide by y or y^2 as it happened in the bernoulli equation and the initial condition says y of 0 equal to 0 are we going through this rigmarole each and every time what you need really is a general uniqueness theorem general uniqueness theorem that can be directly employed off the shelf you pick up a uniqueness theorem importance of proving such uniqueness theorem is now slowly dawning upon us and how do we prove such uniqueness theorems the point that i want to make is that these examples that we have just worked out are rather typical models of reasoning which will give you a proof of the uniqueness theorem idea is to obtain a differential inequality and then proceed exactly as we solve a linear differential equation and we shall not prove the general existence theorem but the key ingredient is the ideas in the following slide

so suppose f of t is a non-negative function on an interval a to b such that f of t is less than or equal to $a + b \int_a^t f(s) ds$ inequality 4.

4 that's a hypothesis a and b are constants and f is non-negative then the conclusion is inequality 4.

5 namely f of t less than or equal to $a e^{bt} - a$ idea of this proof is exactly similar call the right hand side of 4.

4 as capital F of t capital F of t is $a + b \int_a^t f(s) ds$ then notice that little f is less than or equal to capital F little f is less than or equal to capital F and capital F evaluated at little a is capital a 4.

7 what do you do next differentiate 4.

6 differentiate 4.

6 what do you get df by dt equal to bf right you use a fundamental theorem of calculus to differentiate 4.

6 df by dt equal to b times little f but b times little f is less than or equal to b times capital f you got a differential inequality you will proceed exactly as you do with a linear equation multiply the differential equation by e to the power minus b into t a is capital a f of little a equals capital a and then incorporate that into your integral you get f of t into exponential of minus b into t minus little a less than or equal to a which is what we wanted to prove so the proof of this exercise is the proof of this little result exactly follows the same pattern and this is a key ingredient in the proof of the uniqueness theorem

so now we shall move on to the next part of this lecture series and the last part of this lecture series the pendulum restlessly continues to swing punctuating the inexorable march of time

so the pendulum keeps swinging and it marks it calibrates the interval of time for you

so let's go back and recall what the pendulum equation was d^2y by dt squared plus g over l sine y equal to 0 that is equation 4.

9 now what we do is that let us multiply this equation 4.

9 by factor of $d y$ by dt what happens you get $d y$ by dt into $d^2 y$ by dt squared okay

so throw in a factor of 2 throw in a factor of 2 you get $2 y$ dash y double dash what is $2 y$ dash y double dash what are you going to get if you differentiate y dash squared if you differentiate y dash squared you're going to get $2 y$ dash y double dash look at the next term sine y into y dash how do you get sine y into y dash you differentiate minus cosine y if you differentiate minus cosine by you will get exactly sine y into y dash

so what happens when you multiply the differential equation 4.

9 by y prime when you multiply the equation 4.

9 by y prime the left hand side of the equation becomes an exact derivative that's exactly what you see in the next display in the slide d/dt of dy by dt the whole squared minus $2g$ by l cosine y is 0 .

so integrate this and you get dy by dt the whole squared minus $2g$ by l cosine y equals e again i use the letter e for the constant of integration there is obvious reason for it identify the terms in 4.

10 that represent the kinetic energy and the potential energy the first term dy by dt squared is somehow related to the kinetic energy and the second term is somehow related to potential energy you might have to multiply it by some constant you might have to add a certain constant the reference potential but essentially equation 4.

10 spells out the law of conservation of energy and we are going to refer to equation 4.

10 as the energy equation we are going to call it the energy equation okay now let us prescribe initial conditions y of 0 equal to 0 and y prime of 0 equals c where c is a positive constant what does this mean it means the pendulum starts from the mean position with an angular velocity c with an initial angular velocity c you give the push you give a slight push to the pendulum from the mean position and the pendulum starts oscillating incorporate these initial conditions in the energy equation right you have got the energy equation 4.

10 put t equal to 0 put t equal to zero dy by dt at t equal to zero is c and cosine y when time t equal to zero is cosine zero which is one

so equation four point ten simply reads c squared minus $2g$ by l equals e

so e is c squared minus $2g$ by l that's what i am saying

so the right hand side of 4.

10 energy equation has been replaced by $c^2 \sin^2 \theta - 2gl(1 - \cos \theta)$

so the next thing to do would be to take the cosine term on the right hand side if you take the cosine term on the right hand side in equation 4.

12 you will get $\frac{dy}{dt}$ the whole squared equals $c^2 \sin^2 \theta - 2gl(1 - \cos \theta)$ into $1 - \cos \theta$ now recall the trigonometric identity $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$.

so you get $\frac{dy}{dt}$ the whole squared equals $c^2 \sin^2 \theta - 4gl \sin^2 \frac{\theta}{2}$ equation 4.

13 now you might look at equation 4.

13 and you will be happy because you got a first order differential equation equation 4.

13 is a first order differential equation our immediate impulse will be to take the square root and say $\frac{dy}{dt} = \sqrt{c^2 \sin^2 \theta - 4gl \sin^2 \frac{\theta}{2}}$ and that's a variable separable equation we are very happy we can separate the variables and we are going to integrate ok you do that you're going to run into an integral that you cannot compute and that integral is the elliptic integral

so the elliptic integral that i mentioned a few lectures ago appears here in connection with a pendulum equation

so we have $\frac{dy}{dt}$ the whole squared equals $c^2 \sin^2 \theta - 4gl \sin^2 \frac{\theta}{2}$.

so since $\dot{\theta}$ is c which is positive the derivative at time $t = 0$ is positive let us take the positive square root of 4.

13 on both sides and we get $\frac{dy}{dt} = \sqrt{c^2 \sin^2 \theta - 4gl \sin^2 \frac{\theta}{2}}$ which has been called 4.

13 prime 4.

13 prime is a variable separable equation and we proceed in the usual way we divide 4.

13 prime by square root of $c^2 \sin^2 \theta - 4gl \sin^2 \frac{\theta}{2}$ and then integrate both sides with respect to t over the interval 0 to t noting that $y(0) = 0$.

so what do we get we would get $\int_0^t \frac{dy}{\sqrt{c^2 \sin^2 \theta - 4gl \sin^2 \frac{\theta}{2}}} = t$ now we put $\sin \frac{\theta}{2} = u$ we get $\frac{1}{2} \cos \frac{\theta}{2} d\theta = du$ or $d\theta = 2 du$ by square root of $1 - u^2$ and the integral in the last slide transforms to $\int_0^t \frac{2 du}{\sqrt{1 - k^2 u^2}}$ into $2 \int_0^t \frac{du}{\sqrt{1 - k^2 u^2}}$ this last displayed integral is an elliptic integral

so we are naturally led to elliptic integral in the study of the pendulum equation since we cannot undertake a study of elliptic integrals we must abandon this line of investigation and we must take a completely different route any attempt at trying to solve this 4.

13 by the method of separation of variables is going to lead you into trouble now we shall leave this elliptic integrals and try to understand the qualitative behavior of the solution without solving the differential equation remember we have seen this in the past the idea of differential equations is that we must try to get information about the solutions without explicitly solving the differential equation

so we shall assume that c^2 is bigger than $4gl$ let us assume that c^2 is bigger than $4gl$ and look at the right hand side of 4.

13 the right hand side of 4.

$\dot{\theta}$ can never become 0 because if c^2 is bigger than $4gl$ since sine is between minus 1 and 1 the right hand side of 4.

$\dot{\theta}$ is obviously never zero and

so the derivative $d\theta/dt$ must always keep the same sign it must always be positive or it must always be negative and initially look at 4.

$\dot{\theta}$ the derivative was initially positive

so it must be positive forever

so the derivative is positive always that means that θ of t must be a monotone increasing function of time the function θ is monotone increasing with respect to t also since the right hand side of 4.

$\dot{\theta}$ let us go back to 4.

$\dot{\theta}$ what are the lowest that 4.

$\dot{\theta}$ can become the lowest the right hand side of 4.

$\dot{\theta}$ gets to is c^2 minus $4gl$.

$\dot{\theta}$ the right hand side of 4.

$\dot{\theta}$ is least when the sine factor is 1 and c^2 minus $4gl$ is positive say call it a^2

so what do we see

so if you put c^2 minus $4gl$ as a squared then we see that $d\theta/dt$ must always be bigger than or equal to a

so θ of t must be bigger than or equal to at

so θ of t is not only monotone increasing with time it is increasing at least as fast as 80 it is greater than or equal to 80 and

so as time progresses the angle is going to keep on increasing and it goes to infinity but it's an angle remember the pendulum is swinging and

so first the angle will go from 0 up to 2π and then from 2π it will go to 4π

so so that what it's saying is that the pendulum is performing circular motions so the pendulum the c is

so large that the pendulum goes over the top completes the circle and it keeps performing circular motions

so after one circuit the angle goes into the interval 2π to 4π after the second circuit it goes into the interval 4π to 6π and

so on

so if c^2 is bigger than $4gl$ the pendulum has too much energy it starts performing circular motions next let us look at what happens if c^2 is less than $4gl$ and here i am going to give you some very simple calculus exercises show that the angular velocity $d\theta/dt$ must become 0 at some point of time and θ must attain a maximum in some point of time that particular maximum value will be the amplitude of the pendulum and the time at which that happens will be a quarter period and

so the period of the pendulum will be 4 times t_{naught} now suppose not suppose it doesn't happen that means that the derivative is never 0 then what happens if the derivative is never 0 then again θ of t must be monotone increasing as before but this time θ of t cannot reach π because if θ of t reaches π then sine squared θ will become 1 at that particular point because θ will be $\pi/2$ and sine $\pi/2$ is 1 and then right hand side of 4.

$\dot{\theta}$ will become c^2 minus $4gl$ but c^2 is less than $4gl$ remember

so the right hand side of 4.

$\dot{\theta}$ has become negative but the left hand side is a square and that's a contradiction

so that cannot happen

so the angle θ of the pendulum the θ of t can never reach π

so we see that y of t is monotone increasing and it doesn't come anywhere close to π

so the it must have some limit as t goes to infinity y of t must have a limit α as t tends to infinity remember a monotone increasing function must either go to infinity or it must have a finite limit this finite limit cannot be π and

so it has to be less than π that's what we have seen but now the differential equation itself says that the derivative must then have a limit because what the derivative $d y$ by dt squared is c squared minus $4 g$ by l sine squared y by $2 y$ has a limit

so sine squared y by 2 has a limit

so the right hand side of 4 .

13 has a limit in other words the derivative square has a limit $d y$ by dt square has a limit

so $d y$ by dt has a limit because it's always positive

so you've got a situation where you've got a function which goes to a limit and the derivative also has a limit but think geometrically think of a function which settles down to a limit as t goes to infinity that means the graph is becoming flatter and flatter

so we expect that you expect that the derivative if it has a limit it must be zero that's exactly the case if f of t is a differentiable function such that f of t and f prime t have finite limits as t goes to infinity then the derivative must compulsively go to zero i just explain this to you geometrically because f of t has a limit basically means the graph is becoming horizontal it is becoming flatter and flatter and

so the derivative if it has a limit it must actually be zero but can you prove it rigorously using calculus i recommend you do it using the mean value theorem let us apply the mean instead of relying on the geometrical intuition let us back it up with rigorous reasoning let us apply the lagrange's mean value theorem on the interval t comma t plus 1 f of t plus 1 minus f of t is going to be for some c between t and t plus 1 but let t go to infinity the c in the mean value theorem is between t and t plus 1

so as capital t goes to infinity the c also goes to infinity and

so we have the equation f of t plus 1 minus f of t equals f prime c the left hand side goes to 0 we know because f has a limit say l

so f of t plus 1 goes to l f of t goes to l

so f of t plus 1 minus f of t goes to 0 therefore the right hand side f prime c must go to zero

so the derivative compulsorily must go to zero

so we are given two different ways of understanding this

so now let's get back to the pendulum we know that y of t the amplitude is monotone increasing it can't come anywhere close to π it must have a finite limit in that case the derivative y prime t must go to 0 as t goes to infinity

so what do we have we have that as t tends to infinity y of t has a finite limit α which is less than π and y prime of t has a limit and this limit as we have seen is 0 .

now where do we go from here observe that y double prime plus g upon l sine y 0 .

now y has a limit y prime has a limit and

so this equation y double prime plus g upon l sine y equal to 0 tells us that y double prime has a limit as t tends to infinity but the calculus lemma now must tell us that y double prime must necessarily go to zero and therefore the pendulum equation will again give us that limit of y of t must be zero which means that α must be zero but then that means the pendulum is stationary it is not swinging at all and that's a contradiction

so now explain why the point t naught in the last exercise must be a local maximum it is easy the first derivative is 0 .

but what is the second derivative go back to the original equation of the pendulum y double prime plus g over l sine y is 0

so y double prime is minus g over l sine y that will be negative

so second derivative is negative

so it's a point is a local maximum

so the pendulum starts swinging it reaches a maximum and then it has to swing back because it can't go beyond that and then one can again argue that it must have a minimum value but what prevents the maximum value being something and the minimum value being something else

so maximum value could be say 60 degrees maybe the minimum value is minus do i know that how do i know that it swings to the same amount to the right as it does on the left look at problem number five problem number five tells you that the extent to which it goes to the right is the same as the extent to which it goes to the left that's because we have said that y of zero is zero and

so the solution must be an odd function here we appeal to a uniqueness theorem which we are not proved the evasive uniqueness theorem we can also solve exercise 6 without using the uniqueness theorem in a slightly different way let us see we use the energy equation okay

so assume that the maximum amplitude is α and the minimum amplitude is minus β

so look at the energy equation $\frac{dy}{dt}$ the whole squared equals c squared minus $4g$ upon l sine squared y by 2 .

but the left hand side is 0 when y takes the values α or minus β because α is a maximum and minus β is a minimum and therefore we get c squared equals $4g$ upon l sine squared α by 2 and c squared equals $4g$ upon l sine squared β by 2 .

so equating the two things we will get sine squared α by 2 equals sine squared β by 2 from which we deduce α equal to minus β which completes the discussion of problem number 6 all right

so that tells you that if c squared is less than $4g$ by l the pendulum exhibits oscillatory motion it goes to the right and then it comes back and it has a minimum and again it starts moving forward and it has it executes oscillatory motion whereas if c squared is bigger than $4g$ by l it executes circular motions

so what happens at the critical value c squared equals $4g$ by l what happens if c squared equals $4g$ by l is it that the pendulum takes forever to reach the top the topmost point the angle π we must investigate that and see whether that happens it turns out that in this critical case when c squared equals four g by l we are singularly lucky we can actually complete the integration of the differential equation

so let's see how to do that

so remember c squared equals $4g$ by l

so go back to equation four point one three c squared equal to four g by l

so the four g by l can be taken common and we get one minus sine squared y by two that is cosine square y by two

so equation 4.

13 greatly simplifies in the critical case

so we got equation 4.

14 $\frac{dy}{dt}$ the whole squared equals $4g$ by l cosine squared y by 2 .

now observe that if y prime of t naught equal to 0 at some finite time t naught the right hand side of the last equation will give you that y of t naught must be π and y prime of t naught must be 0 but notice that the constant function y

of t identically π also satisfies the pendulum equation and it also satisfies the initial conditions 4.

15 and

so once again the evasive uniqueness theorem that we are not even stated can be appeal to and we see that our solution was constant solution but that is not the case because if it is a constant solution the derivative must be 0 we are assuming that the c is $4g$ by l

so we conclude that y' never vanishes and we may take the positive square root of both sides of 4.

14

so we have the following variable separable equation for the critical case dy/dt equals twice root g by l $\cos y$ by 2 and y of 0 equal to 0 .

so let us try to solve equation 4.

16 by separating the variables

so bring the $\cos y$ by 2 on the left hand side and integrate you get $\log \secant y$ plus $\tan y$ you get $\log \secant y$ by 2 plus $\tan y$ by 2 equals root g by l t denote the right hand side by s that root g by l t you simply denote it by s and just for simplicity you put y by 2 equal to θ then the last equation can be written as $\log \secant \theta$ plus $\tan \theta$ equal to s or $\secant \theta$ plus $\tan \theta$ equals e to the power s

so $\secant \theta$ plus $\tan \theta$ equal to e to the power s is equation 4.

17 take the reciprocal take the reciprocal we get $\secant \theta$ minus $\tan \theta$ equals e to the power minus s adding and subtracting we get $\secant \theta$ equal to $1/2$ of e to the power s plus e to the power minus s e to the one half of e to the power s plus e to the power minus s will be denoted by $\cosh x$ \cosh is the hyperbolic cosine of s and then what is the $\tan \theta$ $\tan \theta$ is one half of e to the power s minus e to the power minus s that is hyperbolic sine

so we got this equation 4.

19 $\secant \theta$ is hyperbolic cosine and $\tan \theta$ is hyperbolic sine

so now we see a relationship between hyperbolic functions and trigonometric functions without having to go into the complex domain by staying in the real domain we pass from the trigonometric functions to hyperbolic functions by a real change of variables this function θ of s given by 4.

19 has a name it is called the gramanian in honor of christopher gourmand this name was given by arthur kaley in 1862 a very interesting history of this good romanian can be found in page 312 of crystals algebra volume 2 which was published in 1900s in fact it's a published before 1900s this is a later edition the inverse of the good romanian appears in cartography and navigation remember that we already encountered cartography when we discuss orthogonal trajectories and again cartography comes in and it comes in cartography and navigation in connection with the mercators projection

so what mericator was trying to construct he was trying to construct a map he was trying to construct a map in which the loxodromes on the sphere are getting mapped as straight lines on the map on your plane map well you will ask me what are lock showed rooms let me explain it to you now loxodromes are curves on the globe with the property that they cut the longitudes at the same angle the curve cuts all the longitudes at the same angle at a constant angle why are such curves important they are important because remember that in navigation concerns ships ships are huge mighty objects when you want to go across the ocean from one point to another point you might think that the best option is to take the shortest path the shortest path is a great circle joining the two points on the surface of the earth but the problem is that when you travel along the great circle the great circle does not intersect the longitudes at the same angle the

angle of intersection keeps changing

so the ship will have to be steered continuously the direction of the ship has to be continuously changed and doing it with a mighty object like a ship is going to be very difficult and very expensive

so the ships do not travel along great circles rather they travel along loxodromes curves which cut the longitudes at the same angle

so now if you take a loxodrome on the surface of the earth what will it correspond on your plane map the map is a planar object it's printed on a sheet of paper and mercator was trying to devise a map in such a way that these loxodromes on the globe correspond to straight lines on the map in an attempt to do that he encountered this inverse of the good romanian function

so i'll give you two references for this one of them is already cited before chapter eight of john mcclary's book geometry from a differential viewpoint i mentioned this book earlier and the second book i'll mention is hl resnikoff and r.

o wells mathematics and civilization very interesting book the second one is a very interesting book he also talks about navigation mathematics and navigation how calculus comes into navigational problems

so here are some exercises on the good romanian show that θ of s is $2 \tan^{-1} e^s$ and show that the good romanian is an odd function and is an increasing function well as far as this problem is concerned there is very little for you to do differentiate this equation θ of s you straight away differentiate the $2 \tan^{-1}$ term that is $2 \cdot \frac{1}{1+e^{2s}}$

so it's an increasing function how do you check that it is an odd function replace s by $-s$ replace s by $-s$ you get $2 \tan^{-1} e^{-s}$ but what is $\tan^{-1} e^{-s}$ it is $\tan^{-1} \frac{1}{e^s}$ but what is $\cot^{-1} e^s$ $\cot^{-1} e^s = \frac{\pi}{2} - \tan^{-1} e^s$ and the $2 \tan^{-1} e^s$ will become $\pi - 2 \tan^{-1} e^s$ and the $\pi - 2 \tan^{-1} e^s$ becomes $2 \tan^{-1} e^s$.

so θ of $-s$ is $\pi - 2 \tan^{-1} e^s$ which is $\pi - \theta$ of s

so θ is an odd function the next problem is very interesting because a similar one appeared in j 2014 the problem that appeared in j 2014 is to integrate a cosecant θ to the power 17 now when you take an odd power of a secant and an odd power of a cosecant you know that if you start integrating by parts you will have to do it over and over again many times and it's not going to be fun

so you want a way to avoid integration by parts over and over again

so how to integrate an odd power of a cosecant or an odd power of a secant the good romanian will help you to do it

so secant θ to the power 17 $d\theta$

so how do you do this put secant θ plus tan θ equal to e^s

so what do you get you get put secant θ plus tan θ equal to e^s

so we get secant θ equal to hyperbolic cosine

so differentiate

so secant θ tan θ $d\theta$ will be equal to hyperbolic sine times $d s$

so secant θ $d\theta$ will be hyperbolic sine s $t s$ divided by tan θ but tan θ equal to hyperbolic sine s $d s$ therefore secant θ $d\theta$ equal to $d s$

so your integral secant to the power 17 θ $d\theta$ will become integral secant to the power 16 which is hyperbolic cos s to the power 16 times $d s$ let

us now incorporate the limits of integration remember the equation $\secant \theta = \cosh s$

so when $\theta = 0$ s would be 0 and $\theta = \frac{\pi}{3}$ would correspond to $s = \log 2 + \sqrt{3}$.

that's easy for you to check

so what do we get the integral would transform to $\frac{1}{2} \int_{\log 2 + \sqrt{3}}^0 \frac{e^s + e^{-s}}{e^{2s} - 1} ds$ and the hyperbolic \cosh is simply $\frac{e^s + e^{-s}}{2}$ and

so you can expand it by the binomial theorem it's much easier particularly when you see definite integrals present you don't have to integrate by parts repeatedly and similarly you can try the cosecant θ to the power 17 here you put $\csc \theta - \cot \theta = e^{-u}$ ok

so i hope you enjoyed this short trip into the world of differential equations and i like to only say a few words to conclude i taken you through many interesting parts of differential equations we saw a lot of geometrical examples we saw examples from physics we saw examples from biology we looked at the orthogonal trajectories we looked at lots of applications we looked at astronomy and stuff like that and i given you many references and i will give you two more references and the first reference is a very beautiful book by gf simmons differential equations with applications and historical notes the second edition is what i am referring to it's published by tata mcgraw-hill third edition has also come out but the second edition will suffice for our purposes this is a very good book it is a delightful reading the historical essays on eminent mathematicians make delightful reading the first 80 pages are accessible to students to the students that they are accessible to you numerous applications of geometry curves of pursuit the branches to crohn problem and orthogonal trajectories are available

so the book is available in indian edition and the second book that i would like to mention is spivak's calculus is a very beautifully written book it's a carefully written book on calculus and on chapter 17 you'll see a derivation of kepler's laws from newton's laws of motion and that was published in 1994

so now our journey has come to an end and i hope you enjoyed this journey into the world of differential equations i'd like to say goodbye but before doing

so it's my pleasure to thank a number of people which i shall do

so now i shall begin by thanking professor nila nataraj the head of the department of mathematics my colleague and the iitb coordinator of this program and i especially thank her for giving me an opportunity for delivering these lectures and more importantly for her constant encouragement and support particularly at times when my enthusiasm was going low she gave me the much needed impetus for going on and then i'd like to thank my colleague professor shantanu day who has listened to these videos worked through the exercises and for his assiduous task of proofreading and for giving me a very big list of corrections which i have incorporated then i'd like to thank professor vikram gadri the head of c deep for offering us this beautiful cd studio with its state-of-the-art facilities and professor niladri chatterjee from iit delhi who is a coordinator of this program i'd like to thank aditya maheshwari the phd student of my department who helped me with those figures and last and most importantly i'd like to express my heartfelt thanks to the technical staff who have been constantly working and who are put up with my pestilence all the time and to them i owe a lot and they are mr tarun negi to all of them i express my thanks and i thank you all my students see you goodbye thank you very much you