

so now we come to the seventh lecture in the series on differential equations and in this seventh lecture we shall discuss uh one important topic linear differential equation and it's close cousin the bernoulli equation

so these are the two types of equations that we shall discuss in this lecture so let's begin

so what is a linear differential equation the linear differential equation is displayed on the slide now it's equation 3.

1 $d y$ by dx plus $p x y$ equal to $q x$ where p and q are functions on x which are defined on a certain interval i assumed to be continuous we assume that p and q are continuous functions defined on an interval i and the differential equation is $d y$ by dx plus $p x y$ equal to $q x$

so let's go ahead and see how to solve this differential equation and this is an important type of equation and in a sense you can solve it completely i say in a sense because although you can write out a formula for the solution of 3. 1 that formula will involve certain integration signs and those integrals have to be interpreted and whether you can solve it completely depends upon your interpretation of what you mean by completely solve it if you if you are satisfied in a formula involving integrals then that's what it is let us now look at the bernoulli equation the bernoulli equation i said is a closed cousin of the linear equation what does it read $d y$ by dx plus $p x y$ equals $q x y$ to the power n which is equation 3.

2 in the slide these two equations go together in fact we shall see that 3. 2 can be reduced to 3.

1 which is why we study them together okay

so let us recall a formula from calculus a very simple formula from differential calculus that you studied and you used it frequently is simply the product rule if you take a differentiable function y of x and you multiply it with e to the power x what is the derivative it is y prime e to the power x plus $y e$ to the power x a straight forward application of the product rule now let us replace e to the power x by something more complicated let us replace it by e to the power ϕ of x where ϕ is a differentiable function and then look at the formula $d dx$ of $y e$ to the power ϕx will be what it will be y prime plus ϕ prime y these two club together into e to the power ϕx again it's a product rule now we select ϕ of x in such a way that ϕ prime equal to $p x$ you see the statement written in red in the slide we select ϕx in such a way that ϕ prime of x equal to $p x$ in other words ϕ of x must be equal to integral $p x dx$ already we have an integral now can you compute this integral $p x dx$ we will come back to that

so select ϕ of x in such a way that ϕ prime of x equal to $p x$ then what do we get we get $d dx$ of $y e$ to the power ϕx equals y prime plus $p x y x$ into e to the power ϕx you see that in the slides

so this slide is all about product rule with specific choices for the two factors

so now look at the last displayed equation in the slide you see the y prime plus $p x y x$ appearing there

so go back to the differential equation 3.

1 you get y prime plus $p x y$ equal to $q x$

so compare 3.

1 with this last equation $d dx$ of y of $x e$ to the power ϕx equal to y prime plus $p x y x$ the whole thing multiplied by e to the power $v x$

so what does this suggest this suggests that this combination y prime plus $p y$ this left hand side of the differential equation 3.

1 y prime plus $p y$ if you multiply equation 3.

1 by a certain e to the power $\phi(x)$ the left hand side will become an exact derivative that's the idea okay

so now what do we do

so this suggests that we multiply our equation 3.

1 by exponential of ϕ of x right

so what are the what was 3.

1 again $d y$ by $d x$ plus $p y$ equals q

so you multiply by e to the power $\phi(x)$ multiplied by e to the power of $\phi(x)$ you get e to the power $\phi(x)$ into y' plus $p y$ on the left hand side but we have just seen that that left hand side will be d/dx of $y e$ to the power $\phi(x)$ that's what we have just seen the left hand side becomes an exact derivative

so you see equation 3.

5

so if you multiply the original differential equation 3.

1 namely $y' + p y = q$ by $e^{\int \phi(x) dx}$ we get equation 3.

5 namely d/dx of $y e^{\int \phi(x) dx} = q e^{\int \phi(x) dx}$ now it is very clear what we should do next we should integrate 3.

5 we should integrate 3.

5 and we get 3.

6 the y has been isolated $y e^{\int \phi(x) dx} = \int q e^{\int \phi(x) dx} dx$ equation 3.

6 in your slide

so now that will give you y of x explicitly namely you divide by e to the power $\phi(x)$ and you recovered the solution of the differential equation y of x

so in a sense we have solved the linear differential equation completely but there are only two problems we need to figure out what the $\phi(x)$ what the $\phi(x)$ is we need to figure out what the $\int \phi(x) dx$ is what is the $\int \phi(x) dx$

so there is an integral sign the $\phi(x)$ can be written in terms of an integral and the second thing is on the right hand side of equation 3.

6 you see one more integration

so we have to perform two integrations we must integrate $p(x) dx$ and get our v after we get our v we must integrate the right hand side of equation 3.

6

so the problem of solving a linear differential equation involves computing two integrals explicitly since the exponential function doesn't vanish equations 3.

1 and 3.

5 are completely equivalent let me remind you what is equation 3.

1 the original differential equation what did we do to the original differential equation we multiplied this by e to the power $\phi(x)$ e to the power $\phi(x)$ is never zero

so you take a equation you multiply it by a non-zero term you get a new equation

so these two equations are completely equivalent

so equation 3.

5 is obtained from the original differential equation by multiplication a non-vanishing quantity we multiply it through e to the power $\phi(x)$ and that is never zero

so the original equation and 3.

5 are completely equivalent

so there is no loss of information and there is no spurious things in introduced into our solution process they are completely equivalent

so now the question is how are we going to find a $\phi(x)$ such that $\phi'(x) = p(x)$ well if we cannot we are out of luck too bad all we can then say is take $\phi(x) = \int a(x) dx + b$ that's all we can see nothing

more look at the statement written in red $\phi(x) = \int_a^x b(t) dt$ well remember p is a continuous function

so the integral $\int_a^x p(t) dt$ exists for any choice of a in the interval and then the final solution would have a very ugly appearance because these integrals will be floating all over the place and we are not able to compute this integral explicitly and that was our problem

so if we cannot find explicitly $\int p(x) dx$ we are out of luck we have to live with it and the final formula will involve integral signs floating around in fact there will be three of them and it will have a very ugly appearance and one cannot do any further with it and the process stops here well

so let us assume that we can find $\int p(x) dx$ let us assume that we can find $\int p(x) dx$ namely let us assume that we are in a position to compute $\int p(x) dx$ what is the solution look at 3.

6 look at the equation number 3.

6 in this slide wherever there is a $\phi(x)$ i am going to replace it by $\int p(x) dx$ then what do you get you get 3.

7

so in 3.

6 replace every occurrence of $\phi(x)$ by $\int p(x) dx$ now there is one important thing that you need to keep in mind namely you have three integration signs appearing in 3.

7 now you have been putting a constant of integration whenever you see an indefinite integral

so you might say that you will put a constant of integration c_1 for the integral appearing on the left hand side and for the two integrals appearing on the right hand side you will put a constant of integration c_2 and c_3

so you will say well there will be three constants of integration floating around no that's not the case the final answer should have only one constant of integration floating around

so the other two constants of integration must somehow cancel out it must disappear remember look at equation 3.

7 exponential of $\int p(x) dx$ on the on the left hand side and exponential of $\int p(x) dx$ on the right hand side where $\phi(x)$ has been replaced by $\int p(x) dx$ remember 3.

7 was obtained by multiplying the differential equation by a certain factor e to the power $\int p(x) dx$ and the $\phi(x)$ was $\int p(x) dx$

so the constant of integration that will appear when you integrate $\int p(x) dx$ should be the same on the left hand side of 3.

7 as well as the right hand side of 3.

7

so the constant of integration that you put for $\int p(x) dx$ on the two sides of 3.

7 must be the same constant and since the edit constant of integration is an additive constant you exponentiate you'll get a multiplicative constant the additive constant c will become a multiplicative constant e to the power c and e to the power c is non-zero and will cancel out from both sides of 3.

7

so two of those constant of integration have disappeared and there will only be one constant left the final integration that you perform with the q term thrown in that is the only constant of integration that will survive in 3.

7 please pay attention to this matter make sure that two of those constants of integration cancel out and the final answer has only one constant of integration

so it is understood that in both occurrences of $\int p(x) dx$ in three point seven are identical and

so the same constant of integration would be assigned for both and that e to the power c would be a factor on both sides and this factor would cancel out so one can altogether neglect putting in the constant of integration when you compute $\int p(x) dx$ because that's going to cancel out anyway

so when you integrate $p(x) dx$ don't bother about putting the constant of integration because e to the power c will cancel out from both sides

so just avoid putting the constant of integration in in the first place when you compute $\int p(x) dx$ however on the right hand side of 3.

7 when you compute the integral with the $q(x)$ term thrown in the outer integral in 3.

7

so to say there the constant of integration is very important the final integration on the right hand side of 3.

7 the constant of integration is very important well all this may sound a little complicated but i assure you that it's not because when we start solving the problems you will get the hang of it very quickly

so don't worry about it it's not as complicated as it appears to be next initial conditions very often you see differential equations coming in with initial conditions the value of the solution at some x equal to x naught is specified in which case the final integration should be done with the definite integral stick

so i advise you not to memorize the formula 3.

7 rather derive it it just takes couple of lines each time you do a problem don't try to memorize formulas rather try to derive this and follow the three steps first compute $\int p(x) dx$ and don't put the constant of integration step number one step number two multiply the differential equation by e to the power $\int p(x) dx$ this is easy step step number three perform the final integration okay and if there are initial conditions prescribed then in this third step employ definite integrals rather than indefinite integrals that's all that there is to it it's very easy there are only three steps and the complication if any is in the computing the integrals

so let's look at some examples some very specific examples for you to digest the matter all right let's go to the first example now solve the differential equation $dy/dx + \tan(x)y = \sin(x)$ equation 3.

8 in the displayed slide

so it is $dy/dx + p(x)y = q(x)$ what is a $p(x)$ function it is $\tan(x)$ so what are we supposed to do we are supposed to compute $\int p(x) dx$ what is the integral $\int p(x) dx = \int \tan(x) dx$ what is $\int \tan(x) dx = \log|\sec(x)| + C$ context there is no need to put the absolute value because secant function is positive on the interval in question

so the absolute value sign can be avoided because secant is positive

so $\int p(x) dx$ is $\log|\sec(x)|$

so e to the power $\int p(x) dx$ is $\sec(x)$ that's easy we have ignored the constant of integration observer

so we are completely ignore the constant of integration as mentioned earlier

so now what are the next step to do multiply the differential equation 3.

8 by whatever we have just obtained namely $\sec(x)$ what happens to the differential equation 3.

8 becomes $\sec(x) dy/dx + \sec(x)\tan(x)y = \sec(x)\sin(x)$ that's the left hand side but that is exactly the derivative of $y \sec(x)$ that's the next display in the slide right hand side of course is $\sec(x)\sin(x)$ that is $\tan(x)$ whatever it is

so what has happened to equation 3.

8 it has transformed to $d/dx (y \sec(x)) = \tan(x)$ after multiplying by x of

integral $p \, dx$ equation 3.

8 goes over to the second last displayed equation $d \, dx$ of $y \secant \, x$ equals $\tan \, x$ simply integrate

so $y \secant \, x$ will be equal to integral $\tan \, x$ now we have to put the constant of integration integral $\tan \, x$ is $\log \secant \, x$ plus c and then you isolate the y x and you write y of x equal to $\cos \, x$ into \log of $\secant \, x$ plus c that's easy the integrations were easy and observe where you ignore the constant of integration in the first step when you're integrating $p \, dx$ we ignore the constant of integration the final integration that we perform involving the right hand side the $q \, x$ thrown in the constant of integration has been put that's 3.

9

so that's the solution of the differential equation let's take an example which appeared in je 2011.

in paper one i have reworded the question slightly and i also changed the notation somewhat

so that is it is in sync with what we are doing here

so what is given to you it's given to you that y of x is a continuous function on the open interval 0 infinity in the original paper it says y is a differentiable function i'm just saying y is a continuous function defined on the open interval 0 infinity and what is it the integral is meaningful because y is continuous

so the symbol $\int_1^x y \, dt$ makes perfect sense now if you take any continuous function g of x and you compute integral of a continuous function from 1 to x the result is going to be a differentiable function

so integral from 1 to x $g \, dt$ is going to be differentiable with respect to x and what is the derivative the derivative is going to be g of x that is the fundamental theorem of calculus that's the fundamental theorem of calculus

so the left hand side of 3.

\int_1^x is the integral of a continuous function from 1 to x

so the left hand side of 3.

\int_1^x is automatically a differentiable function

so left hand side of 3.

\int_1^x is differentiable therefore the right hand side is also differentiable and the x cubed term on the right hand side is obviously differentiable

so the first term $3 \, x \, y \, x$ is differentiable therefore y of x is also differentiable there is no problem

so you do not have to say in the hypothesis that $y \, x$ is differentiable it is enough to say that y of x is continuous because equation 3.

\int_1^x would force y to be differentiable next we we should differentiate 3.

\int_1^x with respect to x and appeal to the fundamental theorem of calculus the left hand side becomes 6 times y of x the left hand side becomes 6 times y of x on the right hand side of 3.

\int_1^x you will get many terms you will differentiate $3xy \, x$ using the product rule and you will differentiate the x cubed term what happens you get a differential equation notice i told you that out of 3.

\int_1^x you are going to produce a differential equation and here it is y prime of x minus y upon x equals x it's a linear differential equation with p of x equals minus 1 upon x if p of x is minus 1 upon x x of integral $p \, dx$ is x of minus $\log \, x$ x minus $\log \, x$ which is one upon x okay

so we must multiply the differential equation by 1 upon x step 2.

step 1 is over step 1 is integral $p \, dx$ that is minus $\log \, x$ computing x of integral $p \, dx$ that is 1 upon x step 2 is what multiply the differential

equation by whatever you just obtained namely multiply the differential equation

by 1 upon x

so what happens to the differential equation $1 \text{ upon } x \ y' - x \text{ upon } x^2$ and that is exactly the derivative of $y \text{ upon } x$

so 3.

11 $\frac{d}{dx}$ of $y \text{ upon } x$ is what the left hand side would become right hand side is 1 because you had an x and you multiplied by 1 upon x and

so the right hand side has become 1.

so 3.

11 is an innocent looking equation you must integrate 3.

11 from 1 to 2.

what do you want the question asks you for the value of y of 2

so you must employ definite integrals now if you stare at equation 3.

10 if you stare at equation 3.

10 there is a specific value of x which is very interesting namely x equal to 1.

so if you put x equal to 1 in 3.

10 what happens to the left hand side the left hand side straight away becomes zero what happens to the right hand side the right hand side becomes three y minus one

so three y minus one is zero

so y is $\frac{1}{3}$

so when x is 1 y is $\frac{1}{3}$

so you got your initial conditions y of 1 equals $\frac{1}{3}$.

so the value of y is one third when x is one you have been asked to find the value of y when x is two

so what should you do you should integrate three point one one from one to 2 if you integrate 3.

11 from 1 to 2 and you get y of 2 by $2 \text{ minus } y \text{ of } 1 \text{ by } 1$ equal to 1 from that you will get the value of y of 2 and that completes the problem it's a pretty easy problem let's move on to the next problem again i had taken a je question which appeared in 2014 in paper the differential equation is $\frac{d y}{d x} + x y$ by x^2 minus 1 equals $x^4 + 2 x$ upon square root of $1 - x^2$ and initial conditions are given to you y of 0 is 0 and the solution of this differential equation with this particular initial condition is denoted by f x what does the question ask you to determine it asks you to determine the integral of f of x from minus root 3 by 2 to root 3 by 2.

so the question is asking you not for the solution but for the integral of the solution over a certain interval again 3.

12 is a linear differential equation it's a linear differential equation what is the p of x p of x is $x \text{ upon } x^2$ minus 1.

remember that in some situations your linear differential equation will not be given in the form 3.

12 what they would give you is they would give you an equation multiplied by x^2 minus 1.

so instead of giving you 3.

12 they may give you $x^2 \text{ minus } 1$ into $\frac{d y}{d x} + x y$ equal to $x^4 + 2 x$ multiplied by square root of $1 - x^2$ if such is the case then you must divide by the coefficient of $\frac{d y}{d x}$ remember when you do these problems it is important to write the differential equation the form $\frac{d y}{d x} + p y$ equal to q there should be nothing in front of d y

by dx if there is something in front of dy by dx if there's some junk in front of dy by dx divide by the junk and isolate the dy by dx term

so it must be first written in the form dy by dx plus p y equal to q there should be nothing else only the dy by dx sitting there make sure that the differential equation is written in that form fortunately 3.

12 is already in that form

so what is p of x x upon x squared minus 1 okay x upon x squared minus 1.

so p of x is

so integral p x dx we have to compute integral p x dx

so i have written it in a slightly different form i multiplied and divided by 2 for obvious reasons and i change the sign of the numerator as well as the denominator again for obvious reasons remember that our differential equation is defined on the interval minus 1 to 1

so x ranges from -1 to 1

so what are we integrating minus $2x$ dx upon 1 minus x squared and the integral is \log mod 1 minus x square then again no need to put the mod because 1 minus x square is positive when x runs from -1 to 1.

so integral p x dx is half \log 1 minus x squared

so what should we compute x of integral p x dx or e to the power integral p x dx what is that e to the power half \log 1 minus x squared what is e to the power half \log 1 minus x squared square root of 1 minus x squared that's what you see in the next slide x of integral p x dx equals square root of 1 minus x squared we must now multiply the differential equation by square root of 1 minus x squared and the left hand side as always will become an exact derivative and the right hand side of equation 3.

12 the 1 upon root x 1 minus x squared will go away because you are multiplying by square root of 1 minus x square and you are simply left with x to the power 4 plus 2 x

so left hand side has become an exact derivative

so you must integrate this equation from 0 to x

so use the fundamental theorem of calculus it will be y root 1 minus x squared minus y of 0 into square root of 1 but y of 0 is 0 remember y of 0 is 0

so that other term the term coming from 0 will will become 0 .

so you simply get f of x root 1 minus x squared equal to f of 0 plus integral 0 to x t to the power 4 plus 2 t dt f of 0 is 0 .

so you get x to the power 5 by 5 plus x squared after that you'll divide by square root of 1 minus x squared of course you're going to divide by square root of 1 minus x squared but notice that you're going to get two terms one term is an odd function the other term is going to be an even function the x to the power 5 is going to give rise to an odd function now when you integrate an odd function from minus root 3 by 2 to root 3 by 2 the answer is going to be 0 integral of an odd function from minus a to a is 0 integral of an even function is twice the integral from 0 to a you know these properties of definite integrals and we should use that

so you will only get integral from 0 to root 3 by 2 x square dx upon square root of 1 minus x squared with a factor of 2 thrown in because it's an even function the easiest way to deal with this integral is to put x equal to sine θ then dx is \cos θ d θ the denominator is also \cos θ the \cos θ term cancels out you simply get 2 sine squared θ d θ how convenient 2 sine squared θ is 1 minus cosine 2 θ and you can easily

integrate and that's the first exercise for you compute the value of this integral

so the g problems are pretty simple it appears to be okay let's move on to the next problem j 2016 paper one

so with some notational changes the differential equation is given on θ infinity and it reads $y' = 2 - y/x$ remember that the $-y/x$ should be brought on the left hand side always rewrite the differential equation as $dy/dx + p(x)y = q(x)$ that's the first thing to be done and i already done that in the solution process i that's the first thing i done is to bring the $-y/x$ to the left hand side and you can see that $p(x) = -1/x$ and $\int p(x) dx = -\ln|x|$ and $e^{\int p(x) dx} = 1/x$

so you are supposed to multiply the differential equation by x and the left hand side becomes an exact derivative it is a derivative of xy right hand side is of course $2x$ okay

so now no initial conditions are given okay

so let us say let us take some convenient point on θ infinity take the point 1 take the point $x = 1$ just for simplicity and let us give some value let us assume that $y(1) = a$ where a is some real number

so we got this equation $d(xy)/dx = 2x$ we got this equation $d(xy) = 2x dx$ integrate it from 1 to x integrate integrated from 1 to x and you use the fundamental theorem of calculus $xy(x) - y(1)$ but $y(1) = a$

so what do you get $xy(x) = a + \int_1^x 2x dx$ from 1 to x
so it's $x^2 - 1 + a$

so the last display gives you $y(x)$ will be $a/x + x - 1/x$

so you divide by x

so what do you get $y(x) = a/x + x - 1/x$ and then you proceed to calculate the limit of $y'(1)$ $y'(x) = -a/x^2 + 1 + 1/x^2$ using the last display in the slide and i would like you to calculate these limits and answer the question now notice that if $a = 1$ then the right hand side simplifies right $y(x) = x$ it will become if a happens to be 1 and x will cancel out and you will get $y(x) = x$ on the other hand if a is not equal to 1 then what happens when a is not equal to 1 then $y(x)$ will be $a/x - 1/x + x$ the $+x$ term is not a problem because it's going to be bounded on θ to 2 but $a/x - 1/x$ what happens to it as x goes to θ as x goes to θ it will either go to plus infinity or it will go to minus infinity depending on the sign of the term $a - 1$.

so if you look at the question in the original paper it says $f(1) \neq 1$ that is $a \neq 1$ now you understand why that exception has been made in the original question paper

so that's why i written it as a comment in this slide that with $a = 1$ the solution is bounded on θ to 2 otherwise the solution becomes unbounded as x approaches θ .

take the next question for the differential equation $dy/dx + 2xy = e^{-x^2}$ is it true that all solutions have a limit as x tends to plus infinity how would you go about doing this it's a linear differential equation what is $p(x)$ $2x$ what is $\int p(x) dx$ x^2 and $e^{\int p(x) dx} = e^{x^2}$

so what are you going to do you're going to multiply the differential equation by e^{x^2} right you're going to multiply the differential equation by e^{x^2} you are going to get $d/dx (ye^{x^2}) = e^{-x^2} e^{x^2}$ of the left hand side

so let's see how to do this problem

so this is a given differential equation $dy/dx + 2xy = e^{-x^2}$ it's a linear differential equation what is your $\int 2x dx$ is x^2

so exponential of $\int 2x dx$ is exponential of x^2 what's the next step multiply the differential equation by e^{x^2} if we do that what are we going to get we are going to get equation 3.

14 prime the left hand side as always will become an exact derivative the left hand side of 3.

4 after multiplication by e^{x^2} will become d/dx of ye^{x^2} the right hand side of course becomes e^{-x^2} what's the next step the next step would be to integrate 3.

14 prime

so we do that we could integrate both sides but there's one small problem the integral on the right hand side cannot be computed in closed form you cannot calculate the indefinite integral of e^{-x^2} explicitly

so what do we have to do we have to settle for definite integrals all right that's all we can do we must use definite integrals

so let's do the following let us integrate both sides of equation 3.

14 prime over the interval 0 to x and what do you get use the fundamental theorem of calculus what do you get when you integrate a derivative give the fundamental theorem of calculus you're integrating you're integrating what you're integrating d/dx of ye^{x^2}

so what you get you will get ye^{x^2} minus the value at 0 of 0 right hand side is the integral from 0 to x of e^{-t^2} dt upon $1 + t^2$ okay a little rearrangement will give you y of x equals y of 0 exponential of x^2 plus exponential of x^2 into integral 0 to x of e^{-t^2} dt upon $1 + t^2$ which is equation 3.

14 double prime which is displayed in the slide now we must pass to the limit as x goes to infinity and see what happens the very first term on the right hand side of 3.

14 double prime which is written in red in this slide y of 0 is a constant and e^{-x^2} goes to 0 very rapidly

so this term the first term y of 0 e^{-x^2} goes to 0 .

now let us take a look at the second term okay what is the second term e^{-x^2} into integral 0 to x of e^{-t^2} dt upon $1 + t^2$ let's see what happens to this we know that e^{-t^2} is less than or equal to 1 for all t but no matter what your t is t^2 is positive

so e^{-t^2} is less than or equal to 1

so what do you get you get the inequality 0 less than or equal to e^{-t^2} by $1 + t^2$ less than or equal to $1 + t^2$

so integrate what do you get integral of e^{-t^2} dt upon $1 + t^2$ which is non-negative is less than or equal to integral 0 to x of $1/(1 + t^2)$ dt everybody can integrate the second integral which is \tan^{-1} of x and everybody knows \tan^{-1} of x is less than or equal to $\pi/2$.

so what do we get we get after multiplying by e^{-x^2} what do we get 0 less than or equal to e^{-x^2} integral 0

to $x e^{-t^2} dt$ by $1 + t^2 \leq \pi/2 e^{-x^2}$ the right most thing $\pi/2 e^{-x^2}$ goes to 0

so by sandwich theorem the integral term in the middle also goes to 0 and our job is done namely both terms on the right hand side of 3.

14 double prime go to 0 as x goes to infinity and we have answered the question that has been posed is it true that all solutions have a limit as x tends to infinity yes not only that not only that we know that that limit must be zero all solutions actually go to zero as x tends to infinity is what we have actually established notice that we have reached a stage where we cannot compute the integrals explicitly integral $e^{-t^2} dt$ upon $1 + t^2$ 0 to x this integral cannot be explicitly computed the final answer has to be written as a definite integral okay now let's come to the this bernoulli equations this is a bernoulli equation of this equation 3.

15 that is displayed in the slide $y' + pxy = qx y^n$ it's a closed cousin of the linear equation px and qx are continuous on the interval I well what is the relation I said we are going to reduce the bernoulli equation to a linear equation first of all if n is 0 then the right hand side is simply qx it's a linear equation already then no need to reduce it if n is 1 then bring the qx term also on the left hand side and write it as $y' + (p - q)x y = 0$ that's a linear equation again

so these two cases n equal to 0 and n equal to 1 are uninteresting because they have been subsumed in the linear case already and the discussion is over

so now to carry the discussion further let us assume that n is different from 0 and 1 .

so let's assume n is different from 0 and 1 and

so now let us divide by y^n and write 1 upon y^n $y' + px$ into y^{1-n} equals qx the right hand side has been isolated now what next if I put u equal to y^{1-n} then 1 3.

15 becomes a linear differential equation let's see how that happens we divide by y^n we divide the differential equation 3.

15 by y^n and what do we get we get 1 upon y^n $y' + px$ equals qx look at the term in red now put u equal to y^{1-n}

so that du/dx equals using chain rule $(1-n)y^{-n} y'$ into du/dx

so you compare the two terms in red in the first displayed equation the second displayed equation obviously from the second equation we are going to substitute into the first equation

so what happens with the differential equation the differential equation transforms into $(1-n)du/dx + pxu = qx$ now you multiply by $(1-n)$ and lo and behold they got a linear differential equation also let me recall you that we have assumed n is not equal to 1 and we assumed n is not equal to 0 because these two cases the differential equation 3.

15 would already be linear and there is no need to transform the differential equation

so we see how to reduce a bernoulli equation into a linear equation all right and a word of warning is seen in this slide in red we are assuming that y of x is not 0 because we are dividing by y^n

so if n is positive we are dividing by a positive power of y and

so if y of x is 0 we are going to be in trouble

so we are going to make an assumption that y of x is not 0 .

suppose you are given initial conditions such as y of x equal to θ then we can't use this method

so you see the bernoulli equation can easily be reduced to a linear equation

so let us take an example let's take an innocent looking differential equation $\frac{dy}{dt}$ equals y into $1 - y$ y of θ not equal to θ solve it with the method of separation of variables and solve it as a bernoulli equation

so what is the equation $\frac{dy}{dt}$ equal to $y - y^2$ the condition y of θ not equal to θ allows us to divide by y^2

so you divide the differential equation 3.

16 by y^2 and you get $-y' \text{ by } y^2 + 1 \text{ upon } y$ equal to 1

so that's a nice innocent looking equation now you put $1 \text{ upon } y$ equal to u if you put $1 \text{ upon } y$ equal to u then $-y' \text{ by } y^2$ is $du \text{ by } dx$

so the differential equation transforms to $u' + u$ equal to 1 whose solution can immediately be done because that's a linear differential equation you solve the linear equation and you get your u what is it u equal to $1 + c e^{-x}$

so you got your u therefore you got your y

so you solved 3.

16 as a bernoulli equation and i think you will agree with me that this is much easier than method of separation of variables let's take two more exercises solve the following homogeneous equations $2xy dx + x^2 - y^2 dy = 0$ equation 3.

17 now i'm asking you to solve this as a bernoulli equation in x observe that 3.

17 is a homogeneous equation you already know how to solve 3.

17 as a homogeneous equation but i am asking you to solve this not as a homogeneous equation but as a bernoulli equation let's see how to do that

so what's the equation $2xy dx + x^2 - y^2 dy = 0$.

so let's write it in the form $dx \text{ by } dy$

so you write it as $dx \text{ by } dy + x \text{ upon } 2y$ equals $y \text{ upon } 2x$ it is the same equation as 3.

17

so and called it 3.

17 prime all right

so you see this differential equation is a bernoulli equation in x it has the form $dx \text{ by } dy + p y^x = q y^x$ to the power n where n is minus 1 in this case

so how do you solve this equation divide by x to the power n and put x to the power $1 - n$ equals u and then proceed do you think solving it as a bernoulli equation is easier than solving it as a homogeneous equations you solve 3.

17 both ways by both methods and you investigate solve the differential equation $\frac{dy}{dx} + 2x \tan y = \sec y$ into e^{-x^2} oh this looks a little scary is it it is but observe that if you multiply by $\cos y$ something happens

so what's the equation $\frac{dy}{dx} + 2x \tan y = \sec y$ e^{-x^2}

so what was proposed multiply by $\cos y$

so when you multiply by $\cos y$ what happens you get this term in red $\cos y \frac{dy}{dx} + 2x \tan y \cos y \sin y$ and from the right hand side the $\sec y \cos y$ becomes 1 you get equation 3.

18 prime now let's put $\sin y = u$ then what is $du \text{ by } dx$ $du \text{ by } dx$ is $\cos y \frac{dy}{dx}$

so the term $\cos y \frac{dy}{dx}$ has also been written in red

so what happens to the differential equation 3.

18 prime it transforms into the linear equation $\frac{d u}{d x} + 2 x u = e^{-x^2}$ and you can you know how to deal with that equation it's a linear differential equation okay
so i think with this slide i shall stop today's lectures you

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