

welcome to the sixth lecture in this series on differential equations

so we shall continue with the study of homogeneous equations from where we left off last time recall that last time we talked about comparison of infinities given two functions

f and g what does it mean to say that f goes to infinity faster than g or slower than g or f and g

go to infinity in the same place we discussed some of these issues last time

so let us take one more

example on this matter

so orders of infinity and differential equations

so let us look at look at

a innocent looking differential equation $\frac{dy}{dx} = y^2 + 1$

into x we are looking at

this differential equation in the first quadrant namely x bigger than 0 and y bigger than 0 .

all

right

so equation 2.

11 that you see in the slide is a variable separable equation

so you might

ask well what's there to do isn't it all easy but let's look at these exercises on equation 2.

11

first exercise is to find the solution of 2.

11 and then to show that the solution cannot escape

to infinity in finite time and then you have to show that the solution tends to infinity at the

same rate as $\log x$ what does it mean to say that the solution tends to

infinity at the same rate as

$\log x$ it means recall the ratio y of x upon $\log x$ goes to a constant limit not equal to zero

limit y of x upon $\log x$ exists and is non-zero and then the next problem is to look at

y of x minus $\log x$ upon $\log \log x$ and you are asked to find the limit as x tends to if

this limit is non-zero then you will say that y of x minus $\log x$ behaves like minus $\log \log x$ or

y of x behaves like $\log x$ plus $\log \log x$ let's try this amusing exercise it's not difficult it might

appear difficult but it's not really difficult the differential equation is a variable separable

equation

so what are you going to do you are going to separate the variables and you will immediately

integrate it namely you will divide 2.

11 by y you will multiply 2.

11 by $1 + y$ and you will

integrate correct and you will be integrating things like 1 upon y 1 upon x etcetera

so you

will get this equation $y + \log y = c + \log x$ it's a very simple looking equation

now from here we have to answer these questions you have to show that y of x

cannot escape to infinity in finite time okay let us look at this in little detail yeah here we are

so we got this equation $y + \log y = c + \log x$ right now suppose y of x escapes to infinity infinite amount of time what does it mean it means that as x tends to a where a is a finite number y of x goes to plus infinity well can that happen if y of x goes to plus infinity then $\log y$ also goes to infinity y also goes to infinity the left hand side of

this equation goes to infinity and the right hand side goes to a constant how is it possible

that one side goes to infinity and the other side goes to a finite limit that will give you a contradiction

so y of x cannot escape to infinity in finite amount of time i am given a slightly different argument but it is equivalent to what i

just said what i have done in this slide is that i have just written the left hand side in a

slightly different form $y \ln(1 + \log y)$ by y right

so y goes to infinity and last time we have seen that $\log y$ goes to infinity slower than y so $\log y$ by y will go to 0

so $1 + \log y$ by y will

converge to 1

so this parenthesis will go to 1.

and y goes to infinity

so the left hand side goes to infinity the right hand side goes to finite limit that's a contradiction

so we

have answered the question immediately we see that y of x cannot escape to infinity

in finite amount of time the next question was what happens as x goes to infinity

what happens as x goes to infinity well first of all notice that the differential equation was $\frac{dy}{dx} = y \ln(1 + \log y)$ let me just write it here for you just $\frac{dy}{dx} = y \ln(1 + \log y)$ right that was the differential equation what

are the and where are we where x is positive and y is also positive

so what are the differential

equation tell you it tells you that the derivative is always positive

so why is monotone increasing

so if you have a monotone increasing function either it must have a finite limit or it must

go to infinity there's no other choice correct

so now one of the next question show that y of x goes to infinity which means we have to show y of x doesn't go to a finite limit look at the

equation here if y of x goes to a finite limit $\log y$ will also go to a finite limit

so $y + \log$

y will have a finite limit c is a constant and x is going to infinity

so $\log x$ is going to infinity

so again you're going to get a contradiction
 so what is it that we have concluded as x goes to infinity compulsorily y must also go to infinity but what the question asks you the question asks you to show that y of x tends to infinity at the same rate as $\log x$ y of x tends to infinity at the same rate as $\log x$ what we have just shown is y of x goes to infinity as x goes to infinity $\log x$ also goes to infinity as x goes to infinity now we must look at the ratio y/x upon $\log x$ and see what happens to the ratio as x goes to infinity well let's look at that we have to look at this ratio limit as x tends to infinity y/x upon $\log x$ apply the l'hospital's rule simply apply the l'hospital's rule you get y' upon 1 upon x or you are looking at the limit as x tends to infinity $x^{\phi'}$ but what is x^y prime go back to the differential equation what the differential equation again it is $d y$ by $d x$ equals y upon x into 1 plus y correct that's a differential equation so what is y' into x y' into x is y upon 1 plus y so x^y prime is y upon 1 plus y plus y one more application of l'hospital gives you one so we see that y of x goes to infinity at the same rate as $\log x$ the ratio goes to one so we say that y of x behaves like $\log x$ in symbols we would write y of x wiggles $\log x$ the next thing that we need to do is to look at limit as x tends to infinity y of x minus $\log x$ upon $\log \log x$ now how do you know that we can apply the l'hospital's rule how do you know that the numerator y/x minus $\log x$ goes to infinity let us look at the next slide what is y minus $\log x$ look at this equation y minus $\log x$ will be equal to c minus $\log y$ c is a constant mind u c is a constant and and we know already that y of x goes to infinity so c minus $\log y$ will go to minus infinity so y of x minus $\log x$ goes to minus infinity so the limit that we're trying to compute is y of x minus $\log x$ upon $\log \log x$ the numerator goes to minus infinity the denominator goes to infinity and so we are allowed to apply the l'hospital's rule okay so we must now apply the l'hospital's rule to the ratio y/x minus $\log x$ upon $\log \log x$ so you differentiate you get y' prime minus 1 upon x in the numerator you will clear the fractions you will get x^y prime minus 1 and you'll pick an x in the denominator when you differentiate the denominator it's going to be 1 upon $\log x$ into 1 upon x so the x will cancel out so what are you left with we are left with computing limit as x tends to infinity x^y prime minus 1 into $\log x$ but what

is $x y'$ prime
 minus 1 go back to the differential equation go back to the differential equation
 $x y'$ prime is y upon $1 + y$ remember
 so that's what you get
 so we are led to computing
 limit as x tends to infinity $x y'$ prime minus 1 into $\log x$ but what is $x y'$ prime remember
 the differential equation $x y'$ prime is y upon $1 + y$
 so what is $x y'$ prime minus 1 y upon
 $1 + y$ minus 1 you can simplify that and you get exactly what is on the slide limit as
 x tends to infinity $\log x$ upon $y + 1$ again it's an infinity by infinity form one
 more application of l'hospital's rule will give you the limit as -1
 so we have completed the
 problem the problem looked a little complicated but i hope you are not convinced that it's
 not we have computed the limit of this ratio y of x minus $\log x$ upon $\log \log x$
 this ratio goes
 to minus 1
 so we can say that y of x minus $\log x$ behaves like minus $\log \log x$ or y of
 x behaves like $\log x$ minus $\log \log x$ perhaps it's a lose way of saying it
 a precise way of saying it is exactly the limit is minus 1 the limit that we
 are computed is minus 1.

so this was an interesting exercise as to how to understand the
 behavior of the solution as x goes to infinity
 so we have got very precise information
 about the growth of the solution y x loosely put we say that the solution y x
 behaves like $\log x$ minus $\log x$ it is hopeless to try and solve this equation
 you might say why do we do this we already have the explicit solution right
 we
 have the solution y plus $\log y$ equals c plus $\log x$ but is this solution as
 explicit as we would like
 to have it it's an equation connecting x and y it's a closed form solution but
 y is given
 implicitly in terms of x and you would like to solve this and express y in
 terms of x but
 such an attempt is going to be pretty useless but without doing that we have
 obtained a very
 precise information about the behavior of y of x what have we done what does
 this exercise show
 it shows you applications of calculus we have computed limits we use
 l'hospital's rule in the
 last example we use calculus to graph the solution the example in the last
 slide has been taken
 from a certain paper of j shackle growth orders occurring in expansions of
 hardy field solutions
 etc it seems to be a very scary kind of a title but there is nothing in this
 paper that is going
 to be of any relevance to you i am putting this reference because this is
 where i got the example
 from and what we have done has nothing to do with the main theme of this paper

and this reference has been put for completeness and correctness rather than for you to look at it you do not look at this paper at all it's not relevant to you clearly understanding the behavior of the solution of a differential equation for large times is important because it is going to give you information about the behavior of the physical system remember where are our differential equations coming from our differential equations are coming from physics they're coming from biology they're coming from chemical kinetics etc so you want to understand what happens to the state of the physical system as time evolves to infinity in other words you want to understand the behavior of the solution for large values of time and we have just seen two simple examples of it using l'hospital's rule and the differential equations one needs to develop theorems general theorems because we would like to get this information without solving the differential equation obviously because in real life you cannot solve the differential equation and then discuss what happens to the solution the differential equation the theory of differential equations is about trying to get information about solutions without actually solving it so one needs to prove theorems about the behavior of solutions of certain classes of differential equations and one such theorem was discovered by ghard you know about ghardi because you have probably seen the movie man who knew infinity he is the person who invited ramanujan to cambridge and ghardy's results was used later by s chandra shekhar in fact as chandrasekhar much later a few decades later applied the theorem of hardy in his study of stellar astrophysics it's very interesting that a mathematical theorem which was proved in the 1910 found applications much later now let us go back to some more exercises on homogeneous differential equations the y equal to bx substitution that we have been talking about so i want to illustrate with two examples from rainville's book i already mentioned rainville's book earlier and we're going to take two more examples from rainville's book example six $y^2 - x^2 \frac{dy}{dx} + x^2 y = 0$ equation 2. 12 rain will give us a slightly different initial conditions i have taken the initial conditions to be $y = \sqrt{3}$ by 2 equals half solve the differential equation sketch the solution curves

and sketch the specific
 one for which y of root 3 by 2 equals half problem seven i'll leave it to you
 to work out
 yourself we will only concentrate on problem number six the first thing to
 note is is
 this a homogeneous differential equation look at the second term minus $x \frac{dy}{dx}$
 the x term
 is homogeneous but look at the first term y minus square root of y squared
 plus x squared
 this one is not homogeneous it is only positively homogeneous and
 so we must stay in
 the first quadrant if we are going to use the y equal to vx substitution and
 the initial
 condition is given in the first quadrant remember carefully chosen
 so rain will ask
 for the solution only in the first quadrant
 so now let us ask what to do in
 the second quadrant how would we solve this equation 2.
 12 in the second quadrant
 so you do this exercise 6 yourself because you have to put y equal to vx and
 complete the
 job it's pretty routine then we already done two three examples of this kind
 but we want to
 understand how to solve equation 2.
 12 in the second quadrant
 so let us see how to solve this
 equation in the second quadrant equation 2.
 12 in the second quadrant how to solve let us blindly
 proceed by the v y equal to vx substitution you just go through the rigmarole
 and you get $x \frac{dv}{dx}$
 plus square root of $1 + v$ square dx equal to 0 and you solve it you divide
 by x and you divide
 by square root of $1 + v$ squared
 so you must integrate dx upon x which is \log absolute value
 of x but now we are in the second quadrant
 so x will be negative what about the other term
 $\frac{dv}{dx}$ divided by square root of $1 + v$ squared $\log v$ plus root $1 + b$ squared
 the quantity
 inside the $\log v$ plus square root of $1 + v$ square is always positive
 so there is no need
 to put the absolute value there but $\log \text{mod } x$ will be equal to \log of minus x
 so when you
 combine the two logs together you get minus $x \frac{dv}{dx}$ but minus $x \frac{dv}{dx}$ is minus y
 so the solution reads
 $\frac{1}{2} \ln |y + \sqrt{y^2 + x^2}| + \frac{1}{2} \ln |x| = c$
 2.
 14 in the displayed
 slide but this is wrong differentiate 2.
 14 then you realize that you do not get back
 the differential equation
 so 2.
 14 is not the solution of the differential equation
 so if
 you proceed blindly you get the wrong answer this is serious you must

understand what goes wrong remember that I carefully told you that if the differential equation is only positively homogeneous stay in the first quadrant and all is well if you move to the second quadrant caution is needed greater caution is needed because if you simply go through the procedure blindly you will get the wrong answer and here is the example if you differentiate 2.

14

what you end up is equation 2.

15 which is not the original differential equation so

now let's ask ourselves where is the flaw the differential equation is only positively homogeneous already told you that and

so the solution procedure as such

is valid only in the first quadrant let us now do the whole calculation afresh let us

not believe in the theory that we are developed let us rederive the whole thing carefully what

is the differential equation $mx + ny$ what is the substitution that we are making y equal

to vx okay if you say y equal to vx what is dy/dx plus $x dv/dx$ this much is okay all right

now let us look at the differential equation m of x comma v x dx plus n of x comma v x dy what

to do that m of x comma v x has been written as m of minus of minus x x has been written as

minus of minus x why have I done that because x is in the second quadrant and so minus x is

positive I can pull out the minus x because the functions are positively homogeneous remember

so I pulled out the minus x to the power k and I'm getting minus x to the power

k into m of minus 1 comma minus v from the second term n I can pull out an x to the

power k because the n term is homogeneous I know that there's no problem and dy is of course $v dx$

plus $x dv$

so now with this one what happens to the differential equation the transformed equation is

the last displayed equation in the slide minus 1 to the power k into m of minus 1 comma minus $v dx$

plus n of 1 comma v into $v dx$ plus $x dv$ equal to 0

so there's a slight difference in the calculation

there are there's minus signs floating around but nevertheless the last displayed equation is

again variable separable

so let's go through it

so proceeding as indicated in the last

slide that is put back what is the m and what is the n from the differential equation we get y minus square root of y squared plus x squared dx minus x dy equal

to 0 you put y equal to bx x is negative

so when x is negative what is square root of y

squared plus x^2 doesn't come out mod x comes out and the second term
 is no problem dy
 is $v dx + x dt$ now this simplifies to square root of one plus $v^2 dx$
 minus $x dt$ equal
 to 0 why is this because mod x is minus x now mod x is minus x and the
 solution of that last
 one is mod x upon v plus root $1 + v^2$ equal to c the solution is mod
 x upon b
 plus under root $1 + v^2$ equal again c is positive because we
 exponentiated
 now you rationalize the denominator in 2.
 16 rationalize the denominator in 2.
 16 and you
 will get the correct solution y plus root of $x^2 + y^2$
 so what are the lesson to
 be learned when the differential equation is only positively homogeneous and
 if you are working in
 the second quadrant be careful now only a mod x will come out in certain
 places and that wherever
 a mod x comes out you should replace it by minus x dot x okay there are some
 final comments
 i'd like to make about these two examples from rainville's book if you look at
 those two
 equations let us look at equation two point one two let's go back to equation
 2.
 12 2.
 12 asks
 you to find a solution which means that either y must be a function of x or x
 must be a function
 of y correct now suppose i ask you to find the solution curve passing through
 a point on the
 y axis take a point on the y axis $0, t$ i want to know whether y is a
 function
 of x or whether x is a function of y in other words it may happen that dy
 by dx may be 0 or dx by dy may be 0.
 the tangent may be vertical or it may be
 horizontal or the slope may not be defined at all because you see both the m
 and the n
 terms are becoming 0 if you look at 2.
 12 m of x, y is becoming 0 and n of x, y is
 becoming 0 at a point on the positive y axis
 so let us look at this a little carefully
 so i want to understand what happens to points
 on the y axis
 so let us take a point $0, t$ and equation 2.
 12 reads dy by dx equals y minus
 square root of $x^2 + y^2$ upon x or i can rationalize the
 numerator and write it
 as minus x upon y plus under root $x^2 + y^2$ squared the first
 expression is a zero by
 zero form if you assume t is positive if you take the upper part of the y axis
 if i take
 a point zero t with the positive then the first expression is a zero by zero
 form look at

the second expression the second expression makes perfect sense because what happens to the denominator remember x is 0 and y is t square root of y squared is $\text{mod } y$ and $\text{mod } y$ is t because t is positive so the second expression puts the value 0 as dy by dx now on the other hand you could write the differential equation as dx by dy equal to x upon y minus square root of x square plus y squared does this expression make sense at points 0 comma t with t less than 0 so if t is less than 0 then what happens at 0 comma t so look at the radical sign square root of x square plus y squared x is 0 so you're left with square root of y squared which is $\text{mod } y$ so you get y minus $\text{mod } y$ so what is y y is t and what is $\text{mod } y$ minus t dx by dy makes sense x upon y minus square root of x squared plus y squared when you put x equal to 0 you get t minus $\text{mod } t$ but $\text{mod } t$ is minus t because t is negative so you may have to write x as a function of y or y as a function of x along the coordinate axis remember that the y equal to v x method fails completely along the y axis because the variable b has no meaning there so here are three more exercises on how to find solutions of homogeneous equations for the y equal to b x substitution find the solution curve of 2. 1 to the same rainville problem through the point 0 comma c we are just discussed that along the y axis the method must completely fail because of v equal to y by x the variable has no meaning so what substitution are you going to use don't say y equal to v x here you take the substitution x equal to v the substitution x equal to v y do it the other way around and the same thing will go through modify the method appropriately so that's how you do problem number eight let us briefly look at problem number nine equation two point one seven dy by dx equal to x plus y by x minus y it's a homogeneous equation y equal to v x substitution of course along the line x equal to y the differential equation makes no sense but away from this line it makes perfect sense the differential equation is homogeneous there is no problem you can get the answer $\log x$ square plus y squared minus 2 tan inverse of y by x equals c the next equation dy by dx minus y by x equals x of y by x again it's a homogeneous differential equation y

equal to v_x substitution will do the trick okay
 so you got large number of exercises
 to do now let us go a little further in the next few slides i am not going to
 do any
 new problems i am going to instead look at this homogeneous differential
 equations and look at
 a little geometry the geometric interpretation behind this homogeneous
 differential equation
 it shows some very interesting features these features are of general interest
 they are not
 going to help you to solve this equation or solve that equation but it is
 amusing to see these
 things geometrically
 so let's take a few minutes and see what's going on you see what is the
 homogeneous differential equation 2.
 $19 \frac{dy}{dx} = f(x, y)$ the f is a ratio of n upon m
 with a negative sign but m and n are homogeneous of the same degree
 so the ratio is homogeneous of
 degree zero f of x, y is homogeneous of degree zero
 so what does equation 2.
 19 say that if
 we take a point x, mx if you take a point x, mx and try to figure
 out the
 slope of the tangent at the point x, mx let's do that
 so let us see what is the
 differential equation 2.
 19 is saying geometrically take a point x, mx take a solution curve
 of the differential equation and take the line y equal to mx take the line y
 equal to mx passing
 through the origin and this line will intersect the solution curve at the
 point x, mx
 so at this particular point of intersection what is the slope of the curve
 look at the
 differential equation $\frac{dy}{dx}$ is f of x, y but the point is $x, m x$
 so we have to look
 at f of $x, m x$ but f of $x, m x$ is f of $1, m$ because f is
 homogeneous of degree zero
 so the conclusion is that the slope of
 the curve the slope of the solution curve at the point x, mx is f of $1, m$
 it only
 depends upon m it doesn't depend upon x
 so at all the points of whenever these
 curves meet this line y equal to mx all the solution curves intersect this
 line at
 the same angle let us look at this like a picture look at the picture the
 picture shows you a line
 y equal to mx and a family of solution curves for the differential equation
 each solution curves
 meets this line at a point the next solution curve meets at a different point
 the tangent meets this
 line at the same angle the angle of intersection the angle of intersection is
 exactly the
 same for all the curves or take another line these curves meet this line y
 equal to mx at these

points these angles are all the same

so all curves of the differential equation cut each line $y = mx$ at the same angle this is not correct

so this is the geometrical meaning of homogeneous differential equations

so now let's go a little further and this angle of course you can easily calculate by using basic trigonometry

so in classical

geometry a family of curves with the property described in the previous slide a family of curves with the property that they all intersect the line $y = mx$

at the same angle such a family of curves are said to be similarly placed similar and

origin is called the center of similitude i shall not elaborate on these words the property that i just described all the curves meet each line

$y = mx$ at the same angle this property is called curves that are similarly placed similar curves the origin as the center of similarity this is very classical

geometry unfortunately it has gone out of vogue but people who do architecture in those

areas they are very much interested in ideas like similarly placed figures and center of

similarity these things will be more familiar to people in architecture rather than mathematics

but there is one property of these curves if this family of curves is described as $\phi(x, y) = 0$

then all other members can be obtained as $\phi(x/c, y/c) = 0$

so if

you take one member of the family and write its equation $\phi(x, y) = 0$ and replace x by

λx y by λy where λ is a constant or replace x by x/c y by y/c

so starting from $\phi(x, y) = 0$ you get another equation $\phi(x/c, y/c) = 0$

in other words look at the slide from equation 2.

21 you get equation

2.

22

so if one of the curves is given by 2.

21 then the other curve will be given by 2.

22

as you keep changing the c you will get the family of all curves

so this is a very beautiful geometrical interpretation of the homogeneous differential equations and

unfortunately this is not to be found in most books i checked many books and i found this in a very ancient book

written in 1913 i just wanted to share it with you and i want to call this the principle of

symmetry in differential equations what does it say it says that if you take one

solution $\phi(x, y) = 0$ a generic solution not one of these exceptional

solutions

sometimes when you solve differential equations y equal to θ is a solution or x equal to θ is a

solution those are some special solutions they are not generic solutions if you take a generic

solution p of x y equal to θ replace x by x upon c replace y by y upon c and you keep changing

the c you're going to get all the solutions

so this is a symmetry i would like to call

it the symmetry in differential equations the principle of symmetry in differential equations so

usually if one knows a solution which is generic then one can obtain all solutions

so what are the

meaning of the word generic that will emerge as we look at examples

so i want to illustrate this

through all the examples that we are seeing we have solved many examples of homogeneous equations

and now we want to apply this and to each of these examples and to understand this geometry behind

this

so of course what i just said will not work in some cases like if y of x is zero if you have

zero solution you replace y by y by c again you get the zero solution you don't get anything

else

so the zero solution will be an exception it will not be a generic solution so

that's how the word usually is being put and i'm not putting this as a theorem because

moment you want to put it as a theorem it has to be stated with utmost precision and i'm

not stating it with utmost precision

so i'm not elevating the statement to a status of a theorem

and i'm not going to prove the theorem either

so for the same reason we shall not discuss the proof of the result quoted about let's look at a different aspect of this symmetry

so let us take

a non-zero real number t and let us put capital x equal to t x capital y equal to t y .

t^2 is called

similarity transformation or homo t^2 .

t^2 is magnification in the x direction by factor of

t magnification the y direction by factor of t it is like enlarging the picture you have a

picture you got a passport size photograph and you want to enlarge it you do the same magnification

the x and the y direction you get a larger picture

so that's a similarity the two pictures

are similar like similar triangles the sides are magnified by the same amount so

that's a similarity transformation

so go back to the differential equation $d y$ by $d x$ equal to

$f(x, y)$ is homogeneous of degree θ .

what happens when I make the substitution 2 .

23 apply the

chain rule $\frac{d}{dt} y = \frac{d}{dx} y \frac{dx}{dt}$ but

what is $\frac{d}{dt} y$

$\frac{d}{dt} y = \frac{d}{dx} y \frac{dx}{dt}$ and then $\frac{dx}{dt} = \frac{dx}{dt}$ but what is $\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{dx}{dt}$ and the 1 upon t will cancel out

so $\frac{d}{dt} y = \frac{d}{dx} y$ is the

same as $\frac{d}{dt} y = \frac{d}{dt} x$ what happens to the right hand side 2 .

24 when you replace x

by x upon t and y by y upon t the t disappears because f is homogeneous of degree θ .

so what happens at new differential equation that you get 2 .

25 is the same as the old differential equation 2 .

24 the differential equations 2 .

24

and 2 .

25 are identical

so we say that the differential equation 2 .

24 is invariant under

similarity transformations or it is invariant under homothety

so the principle of symmetry

says that if $\phi(x, y) = 0$ is a solution then $\phi(cx, cy) = 0$ is also

a solution and all solutions can be obtained from a single solution provided you choose a

generic choice of solution stated differently homothety takes a solution to a solution

or the set of solutions is invariant yeah these things are not exactly relevant to

the exam perhaps but it is educative and I think one should know this

so let's look at

this through the example that we already studied

so let's take the first differential equation that

we integrated $2x + y \frac{dy}{dx} = x^2 + y^2$ equation 2 .

4 we studied

in this chapter what is the solution we got $x^2 + y^2 = c$

equal to 0 take $c = 1$ take $c = 1$ and let's look at the a special

solution $x^2 + y^2 = y$ let's see

so $\phi(x, y) = x^2 + y^2 - y$ will

be what it will be $x^2 + y^2 - y = 0$ plus $y^2 - y = 0$ minus $y = 0$

so $\phi(x, y) = x^2 + y^2 - y = 0$ gives $x^2 + y^2 - y = 0$

0 you see we got all the solutions we took a specific solution $\phi(x, y) = x^2 + y^2 - y = 0$

plus $y^2 - y = 0$ this special solution we took replaced x by

x upon c and y by
 y upon c and we got all the solutions
 so you see we have given the illustration the principle
 of symmetry to the next example $2x + y + dx^2 + x^2 - y^2 = 0$
 what was the general solution we obtained $3x^2 - y^3 = c$
 take c equal to 1.
 you can take c equal to 5 if you like doesn't matter but if you take c equal
 to 0 it will not work
 so when you take c equal to 0 you get an exceptional solution you get a
 solution which is not generic
 so take c equal to 1 you get $\phi(x, y) = 3x^2 - y^3 = 1$
 so all solutions are obtained by replacing x by x upon c y by y upon c let us
 take the third example another example that we solved was $y + dx + x \ln y - \ln x = 0$
 for which we got the solution $y - 1 - \ln y + \ln x = 0$
 which was equation two point one zero
 so again we have got a $\phi(x, y)$ what is $\phi(x, y)$ here $\phi(x, y)$ is $y - 1 - \ln y + \ln x$
 so let us see whether we replace x by x upon c and y by y upon
 c and see what happens let's do that
 so what is our $\phi(x, y)$ in this case
 $3 \ln x + y - 1 - \ln y$ in this case is $y - 1 - \ln y + \ln x$ here of course we
 have to work in the first quadrant but never mind the principle works even the first
 quadrant
 so $\phi(x \text{ upon } c, y \text{ upon } c)$ is what will be $y \text{ upon } c - 1 - \ln(y \text{ upon } c) + \ln(x \text{ upon } c)$
 which is $\frac{y}{c} - 1 - \ln \frac{y}{c} + \ln \frac{x}{c}$
 so when you put $\phi(x \text{ by } c, y \text{ by } c)$ equal to 0
 you get $\frac{y}{c} - 1 - \ln \frac{y}{c} + \ln \frac{x}{c} = 0$
 you check that this is the general solution
 take the next example from Rainville's book $y - \sqrt{x^2 + y^2} + dx^2 - x^2 - y^2 = 0$.
 we got a solution $\phi(x, y) = y + \sqrt{x^2 + y^2} - 1$ but only in the first quadrant since the differential
 equation is only positively homogeneous consider the homogeneous equation $x + y + dx^2 - x^2 + 4y^2 + 4xy = 0$
 taking two more examples from Rainville's book observe that if you take ϕ
 of x, y equal to $y + x$ then $\phi(x, y) = 0$ is a solution namely $y = -x$
 is a solution but then you're not going to get all the solutions from here
 because if you take $y + x$

x and if you replace y by y by c and x by x by c then phi of x by c comma y by c equal to

0 is again the same solution

so you get the same solution you don't get anything

new does this contradict the principle of symmetry no solve the equation 2.26 in

general and identify the special solution well 2.

26 is a homogeneous equation make the y

equal to vx substitution get the general solution the general solution is displayed in red in this slide in this solution if you put c equal to 0 what do

you get you get y cube into x plus y equal to 0 so that's why you see x plus y is a special solution

and now you understand why it is really special a last example you look at the

differential equation $x^2 dy - 4x^2 dx + 2y^2 dx + 7xy dx$ equal

to 0 we see that phi of x y equal to 0 furnishes a solution for each of the choices x plus y and 2

x plus y namely y equal to minus x is a solution and y equal to minus 2x is also a solution if

you start with these solutions and do the x by c y by c trick you will not get all the solutions

you will get the same solution again

so these two solutions are not generic why are they not generic solve 2.

27 in the usual way y equal to vx substitution get the general solution and see what happens solve the differential equation completely and try to figure out what's going on

and the answer is given to you it's $x^2 y + 2x^3 - cx - cy$

so what happens

when you put c equal to 0 when you take c equal to 0 you get $x^2 y + 2x^3$ equal to 0

so y plus 2x is a very special solution okay

so y plus 2x is a special solution but why

is x plus y a special solution look at the last display last line in the display you divide by

c you divide by c then what happens let's see you see the differential equation is $x^2 dy - 4x^2 dx + 2y^2 dx + 7xy dx$ equal to 0.

okay i'm saying that y equal to minus

x and y equal to minus 2x are both exceptional they're special solutions not generic why am i

saying this you have to do this as an exercise it's an exercise for you solve the

differential equation completely solve the differential equation using of course the y equal to v x substitution okay you get the solution the solution

reads $x^2 y + 2x^3 - cx - cy$ equal to 0 that's a general solution now

squared into y plus 2x minus c into x plus y equal to 0 that's a general solution now

when you take c equal to 0 that gives you y plus 2x as a special solution but why is this

one a special solution you see you divide by c and let that c go to infinity
and what do you get you
get x plus y equal to 0
so you see that y equal to minus x is also a special solution
so there are
these exceptional solutions which are not generic and they will always be
there but if you
leave out these exceptional solutions and pick up a generic solution and
replace x by
 x upon c and y by y upon c then ϕ of x upon c comma y upon c equal to 0 will
generate all
the solutions this is a very beautiful part of homogeneous differential
equations
unfortunately this is not discussed in books
so i thought i must really point out
this intrinsic geometry that is present in this chapter i think that this will
close this lecture
and this chapter also next time i am going to start the new chapter on linear
and bernoulli
equations and after that we will probably take a concluding lecture and finish
the
series of lectures thank you very much you