welcome to the fifth lecture in this series on differential equations today we will do a chapter on homogeneous equations

so let me let's first begin with the requisite definitions as to what is a homogeneous domain we'll also define what is a positively homogeneous domain we look at some simple examples we look at homogeneous functions positively homogeneous functions and finally homogeneous differential equations and how to solve them all right

so let's begin with the definition you see it on the slide a subset d of the plane we're always going to be looking at subsets of r2 a subset d of the plane r 2 is said to be homogeneous if whenever you take a point x y in the domain d and multiply it by scalar t t not equal to 0 the point t x comma t y must again be in d in other words you see when you take a point x y in the domain t times x y is simply a point collinear to x y with the origin it is just a scaling of the point x y by factor of t the scaled point must also be in the domain

so you see that all and the scaling factor could either be positive or it could be negative

so as t varies over the real numbers what happens to the set of points t x comma t y it's a line

so a homogeneous domain is a union of lines except possibly for the origin origin may be removed because remember the definition we are requiring that x y belongs to d implies t x t y belongs to d for t not equal to 0.

so this is the algebraic definition of a homogeneous domain now let us see a few examples of homogeneous domains but before that let us look at the picture of a homogeneous domain yeah here you see a homogeneous domain shaded in yellow it is the region between two lines y equal to x by two and y equals two x the region between these two lines has been shaded yellow in the first quadrant and the third quadrant now this union of these two pieces yellow pieces is a homogeneous domain why is it a homogeneous domain we can we can see that if you take a point x y in the yellow region multiplied by t your t times x y is also there

so you see this is a point x y you multiply it by t you come to the collinear point t x t y if t is bigger than zero and you come to this particular point on the opposite side if t is less than zero

so this is true for each such point

so this domain is a homogeneous domain now let's go further often we are not interested in a homogeneity with respect to all real numbers but only with respect to positive real numbers

so likewise let's define a positively homogeneous domain when is a domain said to be positively homogeneous if whenever x y belongs to d t times x y that is the point t x comma t y also belongs to d but this requirement is only for t positive in the earlier case for homogeneous domain we want t times x y should belong to d for all t not equal to 0 this time we are requiring it only for positivity

so let us call such a domain positively homogeneous domain

so we got homogeneous domains and we got positively homogeneous domains

so now let us look at an example of a domain which is positively homogeneous but not homogeneous it is very easy to construct that domain look at the here is a picture the open first quadrant d equals the set of all points x y in the plane such that x is positive and y is positive the open first quadrant why do i say its open first quadrant because the boundary rays the par the parts of the coordinate axis which bound this quadrant are not included in the domain the domain d is the set of all points x y such that x is positive and y is positive

so x equal to 0 is not included

so it's the open first quadrant this open first quadrant is obviously positively homogeneous right if i take x comma y in the open first quadrant then t x comma t y will also be in the open first quadrant for t positive but not for t negative for example 1 1 for example the point 1 1 is in the domain but take t equal to minus 2 and minus 2 minus 2 is not in the domain

so this is not homogeneous but it is positively homogeneous

so i hope you understand the distinction between a homogeneous domain and a positively homogeneous domain both concepts will be used repeatedly in what follows let us look at some examples of homogeneous domains and positively homogeneous domains the entire plane is obviously a homogeneous domain take a point x y in the plane t x comma t y is also in the plane next example is that the plane with the origin removed i remove the origin from the plane

so the punctured plane is it a homogeneous domain take a point x y both cannot be zero both x and y cannot be zero

so either x is not equal to zero or y not equal to 0 multiplied by t where t is any real number which is not 0 then t x comma t y cannot both be 0 because t is already not 0 and we know that x or y is different from 0

so this domain r2 minus the origin is a homogeneous domain the first quadrant as we have seen is a positively homogeneous domain which is not a homogeneous domain instead of that now let us take the next example let us take the first quadrant and the third quadrant like that yellow similar to the yellow regions in the very first slide that we took those two those two pieces those two yellow pieces that we took the first quadrant and the third quadrant that is homogeneous domain now let us take the function f of x y equals logarithm of mod x minus y by x plus y where is this function defined this function is not defined along the line y equal to x and this function is also not defined along the line y equal to minus x

so take the plane remove these two lines and then you get four pieces the union of those these four pieces is a homogeneous domain if you take a point x y and multiplied by t t naught equal to 0 again the point t x comma t y will be in one of these four pieces the last one is a little exercise for you examine whether the set of points in the plane x y such that x bigger than y you would like to call it the open half plane is this open half plane homogeneous is it positively homogeneous ok think about that its not difficult

so these are some examples of homogeneous domains now let us come to the main point of this chapter homogeneous functions and homogeneous differential equations

so when is a function said to be homogeneous first of all it should happen that the domain should be a homogeneous domain that is whenever x y is in the domain of the function t x comma t y must also be in the domain of the function otherwise the definition would not make sense that's exactly why we defined the homogeneous domains before embarking upon defining a homogeneous functions

so a function f of x y is said to be homogeneous if f of t x comma t y equals t to the power k times f of x y

so the right hand side and the left hand side are both defined because whenever x y is in the domain t x comma t y is also in the domain this k is called the degree of homogeneity this k is called the degree of homogeneity and we will say that f is homogeneous of degree k

so again let us take the number of examples x squared plus y squared minus 7 x y it is homogeneous of degree 2.

if you replace x and y by t x n t y you see the factor of t squared coming out f of t x comma t y will be t squared times f of x y is directly possible for you to check this now let's take the next example tan inverse of y by x now this function is not defined when x is 0

so it is not defined along the y axis

so remove the y axis and then you see that when you replace x y by t x comma t y nothing changes tan inverse of y by x is the same as tan inverse of t y by t x in other words f of x y equals f of t x comma t bar

so this is a function which is homogeneous of degree zero this is homogeneous of degree zero let us take the third example square root of x cube minus y cube now we are looking at real valued functions only

so from the domain we have to remove points like 1 comma 2 because when i take x equal to 1 and y equal to 2 the quantity under under the square root becomes negative we do not want that

so what is the domain of this function the domain of this function is a set of all points x y and r two such that x is bigger than y it is a half plane

so this domain is positively homogeneous but not homogeneous

so you say that this function is positively homogeneous of degree 3 by 2 what is the positively homogeneous function the equation f of t x comma t y equal to t to the power k times f of x y should hold for t positive there is an equation the displayed equation f of t x comma t y equal to t to the power k times f of x y must hold for positive values of t

so you say that this function is positively homogeneous

so the last example that you see in the slide square root of x cube minus y cube divided by x square plus y square to the power 3 by 2 you replace x and y by t x comma t y and then you see that its homogeneous of degree zero but t has to be positive

so you got examples of functions which are homogeneous and functions which are only positively homogeneous

so what kind of domains are we going to look at for functions we are going to be looking at functions f of x y m of x y n of x y three functions frequently occurring in differential equations m x y d x plus n x y d y equal to 0 will be the differential equation right

so m is a function of x and y and n is a function of x and y what are the domains for m and n the domains will have to be either homogeneous or positively homogeneous

so what kind of domains are we going to be looking at mostly the entire plane many examples will have the entire plane as the definition open half planes like the last example that you saw square root of x cube minus y cube appears

so you need to look at half planes the open first quadrant this is a very important example the open first quadrant and finite intersections of open half planes described in item number two above

so these are the kinds of domains that we're going to be looking at mostly r 2 minus the origin r 2 the whole plane the whole plane minus the origin a half plane a quadrant

so such domains we are going to be looking at they are all homogeneous or positively homogeneous now let us take up the definition of a homogeneous differential equation assume that m of x y and n of x y are both defined on a homogeneous subset d of the plane and m of x y and n of x y are homogeneous of the same degree that is m is homogeneous m is homogeneous of degree 2 you want n also to be homogeneous of degree 2.

if m is homogeneous of degree minus 3 then n must also be homogeneous of degree minus 3.

so the degree of homogeneity of m and n must be the same then you say that this differential equation m x y d x plus n x y d y equal to zero is a homogeneous differential equation

so let us look at two examples of homogeneous differential equations the first

one is x squared minus y square dx plus 2 x y d y equal to 0 observe that the m here is x squared minus y squared the n here is 2xy they are both defined on the entire plane and they are both homogeneous of degree 2.

 so the first differential equation is homogeneous what about the second differential equation the second differential equation is log x squared minus log y squared immediately if x is 0 or if y is 0 there is a problem
 so we have to remove the x axis and we have you remove the y axis we are getting all the open quadrant the open first quadrant second quadrant third quadrant and fourth quadrant the union is a homogeneous domain
 so replace x by t x and y by t y what happens to log x squared minus log y squared it is log of x squared by y squared and then you've got the second term tan inverse of y by x when you replace y and x by t y and t x respectively the t cancels out
 so both terms are homogeneous of degree zero
 so the second equation is also a homogeneous differential equation as per this definition the following equation would not be homogeneous let us look at equation 2.
1 x plus y dx minus square root of x square plus y square d y is not homogeneous why is it not homogeneous look at the first term m of x y m of x y is x plus y it is homogeneous in the entire plane of degree one what about square root of x square plus y square replace x by t x and y by t y what happens t x squared plus t y squared under root is under root t squared into under root x square plus y squared but under root of t squared is more d f of t x comma t y is mod t times f of x y
 so the function square root of x square plus y squared is not homogeneous it is only positively homogeneous
 so the n x y term here is not homogeneous it is only positively homogeneous you would like to call this differential equation as a positively homogeneous differential equation naturally you wouldn't want to call it a homogeneous differential equation you would like to call it positively homogeneous because it is homogeneous only when t is positive let us take another example square root of x squared plus y square dx plus square root of x squared minus y square d y equal to 0.

 so let us say that this differential equation is taken on x bigger than y
 so half plane x bigger than y not only we take x bigger than y we also take x and y to be positive
 so in this 2.
2 we shall add a further condition that x bigger than y bigger than 0 okay
 so let us make that additional assumption then x square minus y squared will be positive if x is bigger than y bigger than 0.

 so the domain on which x squared minus y squared is defined is positively homogeneous
 so the second term is not homogeneous it is only positively homogeneous the first term is not homogeneous it is only positively homogeneous the second term fails to be homogeneous for two reasons and the first one fails to be homogeneous for a single reason namely more t comes out in the second case not only does mod t come out the domain on which this term is defined is only positively homogeneous i'd like to draw your attention to the fact that most books will call 2.
2 also as a homogeneous equation we are being slightly careful we are distinguished between a homogeneous equation and a positively homogeneous

equation

so why do books call 2.

2 also as homogeneous reason is very simple whenever we are in situations like 2.

2 we simply assume that everything is happening in the first quadrant

so that's a tacit assumptions that books make which is why they don't get into any trouble

so in the first quadrant the method that we will apply for homogeneous equations is identical to the method which we apply to positively homogeneous equation as far as the method of solution is concerned they are the same if we restrict ourselves to the first quadrant but if you try to solve equations which are positively homogeneous but not homogeneous in a quadrant other than the first quadrant you need to exercise some care and we look at some examples as we go along

so in short the method that follows works for positively homogeneous equations as well as homogeneous equations in the first quadrant whereas if it is a homogeneous equation then you do not have to worry about the first quadrants valid throughout the domain

so that's the reason why books don't distinguish between them because they will assume that when the equation is not homogeneous and only positively homogeneous they assume tacitly that we are working in the first quadrant we have been a little cautious paying attention to details just a disclaimer just to keep things simple we shall assume unless otherwise stated that if the differential equation is only positively homogeneous then the domain is either the first quadrant or a positively homogeneous subset of the first quadrant all right

so now we come to the celebrated y equal to vx substitution right in the first lecture i mentioned this y equal to vx substitution the idea goes back to leibniz around 1693 you will see this name appearing right in the first lecture and you will see the substitution y equal to x being mentioned there

so theorem one the substitution y equal to v x transforms a homogeneous differential equation into a variable separable equation that is the reason why homogeneous equations are nice because through a very simple substitution you can reduce it to a variable separable equation the same holds also for a positively homogeneous equation as long as we stick to the first quadrant let's look at the proof y equal to vx

so apply the product rule d y by d x equals v plus x dv by dx simple application of product rule gives you equation 2.

3

so substitute in into the differential equation take the differential equation to be m x y plus n x y into d y by d x equal to zero what is the assumption m and n are homogeneous of the same degree k

so put y equal to v x in this equation what do we get we get m of x comma v x plus n of x comma v x into the d y by d x has been replaced by v plus x d v by d x well m and n are homogeneous

so m of x comma v x will be equal to x to the power k times m of 1 comma v n of x comma v x will be x to the power k times n of 1 gamma with the x to the power k becomes a common factor from both terms and it comes out and i am going to divide by x to the power k and a little rearrangement will give you a variable separable equation that's the proof what happens if the differential equation is only positively homogeneous then in that case remember we work in the first quadrant and the first quadrant x is positive and again m of x comma v x will be x to the power k times m of 1 comma v

so the proof simply goes through as long as we are in the first quadrant that's what i mentioned

so the method of proof goes through identically in the first quadrant for both cases without further ado let us move on to the examples the first example 2 x y d x minus x squared minus y squared d y equal to 0 equation 2.

4 if you go back to the previous lectures you will see that we have encountered equation 2.

4 equation 2.

4 is a differential equations for the family of circles touching the x-axis at the origin and i said you don't solve the differential equation right now we have to wait until we move on to the chapter on homogeneous equation and here we are now we are in a position to solve this differential equation to 0.

4 we employ the substitution y equal to vx and we use equation 2.

3 2.

3 says dy by dx is v plus x dv by dx it is convenient to rewrite 2.

3 in a more easier form in a much easier form d y equal to v d x plus x d v just write d y as v d x plus x d v y just for simplicity it's much easier to do
 so at least in writing is easier

 so we simply substitute y equal to vx this becomes a 2x squared vdx x squared minus y squared becomes x squared into 1 minus v squared and then the d y is v d x plus x d v the x squared cancels out and we are left with 2 v dx minus 1 minus v squared v dx minus 1 minus v squared x d v simplify you get 2.

6 look the differential equation is variable separable v cube plus v d x plus v squared minus 1 x d v equal to zero it is a very easy to solve this differential equation simply divide by v cube plus v simply divide by x what do you get you get d x upon x plus v squared minus 1 upon v cube plus v dv equal to 0.

 so it is a very easy matter to solve this differential equation by employing partial fractions you know how to do that
 so i leave it as first exercise do the routine partial fractions calculations and obtain the solution of 2.

6 is v squared plus 1 into x upon v equal to c and then what is what is v v is y upon x

 so put in the value of v and you get this equation x square plus y squared equal to c y the orthogonal trajectories are circles touching the x axis at the origin that is what we expect from geometrical considerations and that was what i said in the last lecture

 so we have done the problem completely now let us move on to the next example orthogonal trajectories again
 so find the orthogonal trajectories of the system of curves x cube minus 3 x y squared equal to c

 so how do you do that differentiate the equation c straight away disappears what do you get 3 x squared minus 3 y squared minus 6 x y d y by d x equal to zero the three factors out and you get a simple looking differential equation the astute student will recognize that x cube minus three x y squared is a real part of z cube where z is a complex number x plus i y and no price is for guessing what would be the orthogonal trajectories i am pretty sure the students would have guessed what the orthogonal trajectories are i won't tell you guess and guess correctly please okay
 so differentiate the c disappears as i said you get the differential equation x squared minus y squared dx minus 2xy dy equal to 0.

 what the differential equation for the orthogonal trajectories go back to the previous lectures and you figure it out it's x squared minus y square d y plus 2xy dx equal to 0 right that's a homogeneous differential equation m is 2xy and n is x square minus y squared it's homogeneous in the entire plane and
 so the y equal to vx substitution has to be employed and you can simply carry

out the details to completion

so please do it and then check whether your guessed correctly

so again we got another example of orthogonal trajectories coming from complex analysis but we applied the theory of differential equations to understand it now more examples let us look at an example in the first quadrant let us look at an example in the first quadrant because it's going to be only positively homogeneous we have you see a log x and a log y present

so automatically x has to be positive and y has to be positive the differential equation is defined only in the first quadrant and the first quadrant is positively homogeneous but not homogeneous but that's okay

so your differential equation is a positively homogeneous differential equation 2.

7 replace x and y by t x and t y respectively and the t you see the things are homogeneous of degree one positively homogeneous degree one

so let us also ask for the solution of this differential equation satisfying the condition y of 1 equal to 1 that is the solution of 2.

7 remember is a family of curves and one of those curves will pass through the point 1 1 and you are asked to find which one of these curves is it

so think of x as a function of y

so think of x as a function of y and we'll write x x of y x is a function of y

so we'll write x of y and y is the independent variable the differential equation is easily seen to be positively homogeneous and you can use the substitution y equal to v x and we will get the vx equation the v dx plus log v into v dx plus x dv equal to 0 the usual routine you go through

so you get a variable separable equation what is the variable separable equation that you get that is equation 2.

8 that is displayed in the slide equation 2.

8 is easily seen to be variable separable separate the variables integrated x when x is 1 y is 1

so what is the value of v v is also 1

so to integrate 2.

8 you could employ definite integrals because definite integrals incorporate the initial conditions in the solution process or you can apply indefinite integrals if you will but you can certainly solve it in whatever way you want i employ definite integrals and i got an easy integration

so you do this integration and you get equation 2.

9 log y minus log of 1 plus log y minus log x is 0.

so one of the logs can be removed by exponentiating you get y equals 1 plus log y minus log x that is equation 2.

10

so 2.

10 describes the solution curve of our differential equation 2.

7 passing through the point one comma one well one could stop here and say okay we have solved the differential equation what more do you want but i would like to know how to sketch this curve how to sketch the solution curve it's of interest to do that

so how do we do that well first of all observe that the differential equation gives the value 0 to the derivative dx upon d remember i said think of x as a function of y and

so compute d x d upon d y d x upon d y will be what look at equation 2.

7 look at equation 2.

7 and compute dx upon d y at the point 1 1 what will happen the log y minus log x term becomes 0

so d x by d y is 0 this is exactly the reason why i said we regard x as a function of y and not the other way around
 so now we got x prime of 1 is 0 the derivative of x is 0 at y equal to 1.

 so it is of interest to know whether this point y equal to 1 is a point of local maximum is it a point of local minimum or what is it when you want to do a curve sketching this point becomes important points of maxima minima points of inflection and stuff like that
 so we must compute the second derivative x double prime of y now you will say your initial impulse would be to say take equation 2.
10 and do an implicit differentiation start from 2.
10 and do an implicit differentiation and compute d2x by dy squared but i urge you not to do
 so instead i want you to use the differential equation as much as possible don't use the solution use the differential equation to the extent possible
 so let us see how to compute the second derivative from the differential equation directly and answer the following questions is y equal to one a point of minimum a maximum or a point of inflection is there a value of y bigger than one at which the derivative vanishes how many points of maxima minima are there if you like can you say something about the function x of y whether it is increasing or decreasing monotonicity properties of this function sketch the graph of the function x of y near the point 1 comma 1 do you think x of y would tend to 0 as y tends to a for some positive a bigger than 1 what happens is y goes to infinity
 so these are the natural questions that you come that you discuss when you want to do curve sketching ok
 so let us go back to the first question how to compute the second derivative directly from the differential equation it is important to do that i will come back to this point again later it is important to get as much information as you possibly can directly from the differential equation without solving the differential equation why because differential equations that arise in real life cannot be solved completely we know that and
 so these are only toy problems that will teach you how to deal with differential equations the real problems are much more complicated
 so we must try to use these toy examples and try to do those things which you will actually do in real life namely without using the solution directly from the differential equation glean as much information as you possibly can
 so let's start with the first thing enquire whether one is a point of maxima or a minima for the function x of y you have to compute the second derivative second derivative where at the point y equal to 1.

 so let us take the first derivative the differential equation dx by d y when rearranged gives you x into log x minus log y upon y go to the differential equation it is written as m dx plus n d y equal to 0
 so get your dx by dy dx by dy will be minus n upon m write that down that's what we are done it is x into log x minus log y upon y
 so we must differentiate this expression with respect to y using the product rule and quotient rule when you use the quotient rule what happens you will be getting a y squared and you are going to get ugly expression you will get an ugly expression of the y square in the denominator but we are computing the second derivative at y equal to 1
 so the denominator will become 1
 so i am only interested in the numerator i don't even have to write the denominator also when you want to compute the numerator of this of the

derivative what happens it is going to be y times the derivative of the numerator minus the numerator times the derivative of the denominator
 so that's what i'm saying it's y times the derivative of the numerator what the derivative to the numerator there are two terms and you use the product rule
 so y x into d d y of log x minus log y minus x into log x minus log y times the derivative of y plus there's a third term plus x prime into y into log x minus log y that term has not been written here in the next line why have i not written is it because i made a mistake no i not made a mistake x prime of 1 is 0
 so that term is going to drop out
 so there's no need to write that term
 so not written the term x prime of 1 because it's going to drop out
 so i have two terms here and x is one and y is one
 so what is d d y of log x it is x prime upon x but again x prime is zero at y equal to one
 so that term also drops out next term ddy of minus log y that is minus 1 because y is 1 and here in this term you put x equal to 1 and y equal to 1 the log x minus log y term disappears
 so the second derivative at 1 is minus 1.
 notice that we are not use the solution at all we are directly worked with the differential equation and we have got the important piece of information that that point y equal to 1 is a point of local maximum and
 so you know how the graph should look like near a local maximum let us start drawing the graph now we have enough information for us to draw the graph here we have the graph notice that we are looking at x as a function of y
 so really we should be doing the graph like this we should be drawing the graph like this but never mind i already have the graph drawn and the horizontal axis is the x axis and the vertical axis is the y axis
 so x is a function of y and
 so you see that the point 1 1 is a local maximum
 so the graph must look like this near the point one one
 so we answer the second question in the exercise draw the graph of the function solution curve near the point one one now let us ask why have i continued the graph like this and why have i continued the graph like this the next question that i want to know is that if y increases according to this graph the x will decrease as y becomes larger and larger the x value comes close to zero and then and then when y increases from zero to one the x value will increase from zero to one in other words x as a function of y increases over here and decreases over here how do we know that first thing to do is that again go back to the differential equation what the differential equation give you the differential equation gives you x prime of y equal to log of y by x upon minus of y by x
 so x prime has no other roots you see we are looking at we are looking at the the first quadrant only x y has a local maximum at y equal to one
 so what are the meaning of saying that it has a local maximum at y equal to one
 so when y becomes larger than 1 the x value becomes smaller than 1 because at x y equal to 1 the value of x equal to 1
 so let me see notice that if x at the value y equal to 1 we know that the x equal to 1 okay
 so y equal to 1 is a local maximum
 so if y increases beyond 1 then x must decrease it's a local maximum for 1 is a maximum value
 so when you when you go beyond the maximum value the value of x is decreased the value of x is decreased that means what x is going to be less than 1 and y is going to be bigger than 1
 so what is log of y by x

so y by x is bigger than 1
so log of y by x is positive
so what happens here
so you see x prime of y x prime of y is log of y by x upon minus of y by x remember that we have just seen that the numerator is positive and we are in the first quadrant
so the denominator is negative
so the derivative is negative beyond 1.

so as a result x y continues to decrease as y increases beyond 1
so y by x
so y by x must be bigger than 1 and therefore log of y by x must continue to be positive and
so the derivative continues to be negative and
so the x of y is strictly monotone on 1 infinity and similarly you can see that it must increase from 0 to 1.

so that is why the graph has been drawn this way now the question is how do i know that this graph goes all the way to the origin how do i know that when y goes to 0 x must go to 0 and how do i know that when y goes to infinity x must go to 0.
here we will have to go back at this stage we will have to go back to equation 2.
10 here we go back to equation 2.
10 and look at this equation y equal to 1 plus log y minus log x suppose if y goes to infinity suppose if y goes to infinity then what happens the left hand side goes to infinity now bring the log y to the left when y goes to infinity what can you say about y minus log y minus 1 what can you say about y minus log y minus 1 that will also go to infinity let me again do this for you on on the sheet of paper what do we have we got y minus log y minus 1 equals minus log x
so when y goes to infinity is it correct to say y minus log y minus 1 also goes to plus infinity how if you believe this that means if
so then minus log x must go to plus infinity
so x must go to zero why is this true why do i say that y minus log y must go to plus infinity because you see when y goes to infinity log y also goes to infinity
so isn't this an infinity minus infinity kind of an indeterminate
so how do i casually say that y minus log y goes to plus infinity we can see that y goes to infinity faster than log y y goes to infinity faster than log y therefore y minus log y will still go to plus infinity let us look at the exercise that i give you on the slides
so to think about this exercise let us take a look at equation 2.
10 y equal to 1 plus log y minus log x note that x must tend to 0 as y tends to infinity the last part 3d as y goes to plus infinity it must happen that x must go to zero plus otherwise the right hand side of two point one zero would go to infinity slower than the left hand side which is a contradiction or in other words bring the log y on the left hand side and you can argue as i argued before
so basically what am i saying we are saying that certain things go to infinity faster than certain other things log y goes to infinity faster than log log y y goes to infinity faster than log y y squared goes to infinity faster than y e to the power y will go to infinity faster than y
so what do you mean by all these things
so here is an exercise discuss this last part precisely what does it mean to say that a function f of t tends to infinity slower than g of t where f and g

are defined on the same interval containing a and t goes to a

so what would be a appropriate definition we could say that f t goes to infinity slower than g t if the ratio ft by gt goes to zero in the present context can you prove that log y goes to infinity slower than y by applying the l'hopital's rule for example by the same logic log t to the power thousand will go to infinity slower than t to the power one upon ten thousand generally no matter how large your n is and how small positive your a is log t to the power n will go to infinity slower than t to the power a as t goes to infinity

so this fact about comparing the growth at which two functions the rates at which comparing the rate at which two functions go to infinity is extremely important in calculus i repeat it is extremely important in calculus to know how fast one function goes to infinity relative to another function or how fast does one function go to zero relative to another function you could have a ratio f upon g both f and g may go to infinity or both of them may go to 0 but which one does faster calculus in essence is the comparison of the growths of functions relative to one another either growth to infinity or decay to zero whichever is the appropriate case it must be noted that given two functions f t and gt the question of whether f t goes to infinity faster or gt goes to infinity faster is usually not easy it is easy in the case of log y and y because you can straight away apply the l'hopital's rule but often you are not going to be

so lucky to apply l'hopital's rule and the problem is going to be pretty hard to decide which one goes to infinity faster or do this two functions go to infinity at the same rate as an example let us take the function f t to be the number of primes in the interval 1 to t

so suppose if t is 10 then f of t is four there are four primes two three five and seven similarly if t is hundred f of t is the number of primes between one and hundred and you know that the number of primes goes to infinity with t the number of primes is infinite

so f of t goes to infinity as t goes to infinity let us look at another function g t equal to t upon log t t upon log t also goes to infinity as t goes to infinity apply lopithal's rule we have just seen that t goes to infinity faster than log t but what about the ratio f t by g t does f t goes to infinity faster than gt or is it the other way around or are they going to infinity at the same pace this question was a notorious conjecture by gaussian independently conjectured that ft and gt go to infinity at the same rate but this conjecture remained unproved for nearly 100 years it was settled independently by two french mathematicians one in 1896 by hadamard and the other one in 1898 with the level poor song and this theorem is called the prime number theorem it's one of the most remarkable theorems in number theory

so you see that comparison of infinities is not an easy task and the idea of comparison of infinity has been beautifully expounded by gh hardy in this little tract called orders of infinity we'll take now another example but we shall probably do this in the next lecture we will take this innocent looking differential equation d y by d x equal to y upon 1 plus y into x this is a variable separable equation and we can actually find the solution again the solution will present itself in implicit form and what we want as we are done in this example we want to understand what happens when x goes to infinity how does y go to infinity we'll take this up next time we'll stop here for today but meanwhile you can solve equation 2.
11 and be ready you