

welcome students to this fourth lecture
 in the series on differential equations
 so let's begin with where we stopped last time
 let's look at the slide that is exercise number 10 where we stopped we take
 the family
 of all circles touching the y-axis at the origin and i asked you to draw
 these
 circles in blue pen and also draw with red pen circles which are orthogonal to
 the blue circles
 and with your picture it's interesting to compare this picture with the
 picture on page 635 of
 rustic and holidays book already cited
 so the question is that does your picture approximate
 this picture on resnick and holiday's book the picture that you see in russian
 holidays book
 is the picture depicting the equipotential lines and the lines of force due to
 an electric
 dipole and the question that i have is as the length of the dipole becomes
 smaller and
 smaller does the picture that you have created approximate the picture on
 resnick and holiday
 better okay i'm not sure whether done the sketching of those circles but
 anyway i will show
 you what i have with me over here
 so let's see yeah
 so there you see the blue circles touching
 the y axis i had drawn four of those and i also drawn four red circles cutting
 the blue circles
 at right angles and these red circles are circles tangent to the x-axis at the
 origin that's
 a nice picture that you have and we'll come back to this two families of
 circles in the
 next part of this lecture series all right
 so now let's get back to where we are
 so let's
 go
 so now let us consider another problem consider two circles with unit radii
 and centers
 $(-2, 0)$ and $(2, 0)$.
 the circles have unit radius the radius is 1 and the
 centers are placed at $(-2, 0)$ and $(2, 0)$.
 you can easily write
 down the equation of these circles right what is the equation of one of the
 circles
 $(x - 2)^2 + y^2 = 1$.
 for the other one $(x + 2)^2 + y^2 = 1$.
 now call the first equation $s_1 = 0$ the expression $(x - 2)^2 + y^2 - 1$ let us call it s_1 for simplicity and the other
 expression
 $(x + 2)^2 + y^2 - 1$ let's call that as s_2 and then let us
 look at
 this equation 1.
 37 in the slide $s_1 + \lambda s_2 = 0$ where λ is a real number and

as

you vary λ you get a family of circles 1.

37

so equation 1.

37 represents a family of circles

and the question that has been asked is to find the first order differential equation

for the family 1.

37 well how do you do it you take equation 1.

37 and you differentiate

it with respect to x you get one more equation and you eliminate λ between these two

equations and you get your differential equation okay sketch the circles 1.

37 with blue pen and

then is there an exception is there a certain value of λ for which this equation is not a

circle there is one value of λ for which 1.

37 is not a circle it is some other curve a limiting case as it were

so what is that limiting case point out the limiting case in your picture

also sketch with a red pen circles that cut the blue circles orthogonally

so again you

get a beautiful picture and in the next slide you see those expressions $s^2 x$ minus 2 squared

plus y^2 minus 1 when you simplify you get x^2 plus y^2 minus $4x$ plus 3.

and what

about $s^2 x^2$ plus y^2 plus $4x$ plus 3 and equation 1.

37 has been written in detail in

this slide and you differentiate this equation with respect to x you get one more equation you

get two equations involving λ you simply eliminate λ and you get your differential

equation for the family can you think of equipotential curves in electrostatics for example

i gave you a hint look at the pictures on page 635 of resnick and holiday's book now

you got another family of circles are these circles equipotential lines

of a certain configuration of charges i'd like you to think about this does your

physical intuition help you to construct a situation does it help you to construct a

setup come up with a setup of charges such that the equipotential lines are exactly given by

this equation x^2 plus y^2 minus $4x$ plus 3 plus λ times x^2 plus y^2

plus $4x$ plus 3 equal to 0.

suppose you do succeed in setting up such a physical system whose eq potential curves are as in exercise 12.

so the equation depicted here are the

equipotential lines what about the lines of force if you look at the pictures in wrestling holidays

book on page 635 he depicts the equipotential lines by solid lines and the

lines of force by dotted lines and what do you see in those pictures on page 635 the dotted lines intersect the solid lines at right angles the lines of force intersect the equipotential lines along right angles so the question is that can you figure out what would be the lines of force the lines of force will be those red circles that you drew and the equipotential lines will be the blue circles that you drew okay maybe as a hint let me give you that you should think of the setup in three dimensions the setup will be in three dimensional space and you take in the three dimensional space it'll be equipotential surfaces and you slice the surface by the $x y$ plane and then look at the picture in the $x y$ plane when you take a surface and you slice it with $x y$ plane you get a curve in the $x y$ plane don't you so you should do the setup in three dimensional space and then slice the picture by the $x y$ plane and look at the cross section i'll give you the answer the answer can be found on page 107 problem 2.

47 of griffith's book introduction to electrodynamics the third edition which appeared in 1999 please pay attention to the edition number of the book okay so this is another example from electrostatics so what are these examples showing you they're showing you two families of curves you got a blue circles a family of blue circles and a family of red circles and the red curves and the blue curves meet each other at right angles before progressing let me give you three exercises on trying to find differential equation satisfied by one parametric family of curves write down the equations for the one parameter family of circles in the first quadrant touching both coordinate axes so what would be the center of the circles c comma c what with the radius c x minus c the whole squared plus y minus c the whole squared equals c squared differentiate with respect to x you get one more equation eliminate c between these two equations and you get your differential equation problem number 15 you look at a parabola y squared equal to $8x$ and at each point on this parabola there is a tangent line and as a point varies along the parabola you get a one parameter family of lines you get the tangent lines to the parabola so you've got again a one parameter family of tangent lines for example you can take a point as a t squared

280.

remember how do you parameter is a parabola and where the t is a parameter which varies along the real line and for each point of the parabola $(8t, 280 - t^2)$ in this case so you find out the equation of the tangent at the point $(t^2, 280 - t^2)$ and you get a one parameter family of lines parameterized by t and find the differential equations for these lines the third problem that i am giving you is look at the system of curves $x^2 + 9 + c + y^2 + 4 + c = 1$.

so what are the parameter here c c keeps varying over the real numbers you get a one parameter family of curves they are conics some of them are ellipses and some of them are hyperbolas of course if i take c to be minus 10 then there is no curve which satisfies this equation

so c cannot be terribly negative so for a certain range of c from say minus 9 to infinity minus 9 excluded you get a family of conics you get a one parameter family of conics a family of ellipses in the family of hyperbolas

this family is called a confocal family and what do you do you have to find the differential equation satisfied by this family of curves

so how do you do it again differentiate the given equation with respect to x what will you get $2x + 2y \frac{dy}{dx} + 4 + c = 0$ the 2 will cancel out what do you get you get the original equation and the new equation regard them as equations for $1 + 9 + c$ and $1 + 4 + c$ the given equation is already on the slide on a slide you have the equation $x^2 + 9 + c + y^2 + 4 + c = 1$ after differentiating with respect to x you get one more equation you've got two equations for $1 + 9 + c$ and $1 + 4 + c$ solve these two equations simultaneously you get $1 + 9 + c = \text{whatever}$ $1 + 4 + c = \text{whatever}$ take the reciprocals $9 + c = \text{something}$ $4 + c = \text{something}$ take the difference c goes away and you get a differential equation that will be the differential equation satisfied by this one parameter family of conics it's a very interesting exercise please do it because we're going to return to this problem very soon okay next thing we do is orthogonal trajectories if you turn to page 635 of resnick and holiday's book as suggested you would be naturally led to the following geometrical configuration of curves in the plane known as orthogonal systems of curves in the plane

so what are orthogonal systems of curves in the plane they are two systems of curves in the plane a set of curves c_1 and a set of curves c_2 each curve in c_1 intersects each curve in c_2 at right angles like your blue circles and your red circles each blue circle meets each red circle orthogonally so two such systems are called orthogonal trajectories of each other so we say that the family c_2 is a system of curves orthogonal to the family of curves c_1 and vice versa the phrase orthogonal trajectories is frequently used in this context we say that the family c_2 is the orthogonal trajectory to the family c_1 and vice versa or we would say c_1 and c_2 are orthogonal trajectories of each other they're orthogonal to each other why should we study orthogonal trajectories who cares about them reason number one they are beautiful geometry is always appealing they are pretty things so that's good enough reason but for those who wish to see realistic examples they appear in fluid mechanics they appear in optics recall that in optics you got the wave fronts and you got the rays the rays are always perpendicular to the wave fronts so wave fronts form one family of objects and the rays form a different set of objects and elements of one intersect the elements of the other at right angles so the so orthogonal trajectories appear naturally in optics orthogonal trajectories appear in mathematical cartography what is cartography it is a science of map making the science of map making is very old and it is very rich and very mathematical the finnish cartographers cartographers from finland they did a phenomenal amount of work they contributed significantly to the science of cartography if ever you have seen a globe have you noticed that the latitudes and longitudes cut each other orthogonally so the family of latitudes and the family of longitudes form a pair of orthogonal trajectories so orthogonal trajectories appear in geography and calculus plays a important role in geography look at chapter 8 of the book of mccleary john mccleary geometry from a differential viewpoint is an amazing book on chapter 8 you see cartography being discussed there are some beautiful historical references and he talks about some of the theorems that go back to euler and gauss the two great masters of the 18th and the early 19th century they have contributed already to cartography another mathematician whose name

appears in the science of cartography is Lambert who is also a mathematician
 now we shall move on to an area within mathematics
 and explain how orthogonal trajectories appear over here and this area of
 mathematics is called
 the theory of functions of a complex variable and this idea is used by
 aerospace engineering
 the theory of functions for complex variables are used by aerospace engineers
 so
 let us look at some simple examples coming from the theory of functions for
 complex
 variables I assure you we will not use anything that you are not familiar with
 already nothing
 that I've talked here will uh will go beyond your 12 standard syllabus
 so don't be scared by
 the theory of functions of a complex variable
 so let us take a function f from \mathbb{C}
 to \mathbb{C} given by $f(z) = z^2$ you separate it into the real and
 imaginary
 parts write $f(z) = u(x, y) + i v(x, y)$ and
 so what are the real part of $f(z)$
 write z as $x + iy$ and you square it what are the real part $x^2 - y^2$
 minus
 y^2 squared over the imaginary part $2xy$
 so now let us look at the system of curves $u = \text{constant}$ and $v = \text{constant}$ in other words let us look at $x^2 - y^2 = a$
 minus $y^2 = b$
 equals a and $2xy = b$
 so again you are taking a function of a complex variable namely f
 of z equal to z^2 $z = x + iy$ square it take the real part $u = x^2 - y^2$
 take the imaginary
 part $v = 2xy$
 so what is $u = x^2 - y^2 = a$ $v = 2xy = b$ what is $v = 2xy = b$ right and set
 u
 equal to constant and set v equal to a constant
 so $u = x^2 - y^2 = a$ and $v = 2xy = b$ what is this
 family of curves $u = x^2 - y^2 = a$ that is what are the curves $x^2 - y^2 = a$
 $y^2 = a - x^2$
 a they are a family of rectangular hyperbolas what is this family $v = 2xy = b$
 that is $2xy = b$
 equals b again you get a family of rectangular hyperbolas I'll leave it to you
 to verify
 that $x^2 - y^2 = a$ cuts the hyperbola $2xy = b$ to $x^2 - y^2 = a$
 orthogonally that is if you take a point (x_0, y_0) which lies
 on both
 curves calculate the slopes of the two curves at the point (x_0, y_0)
 the product
 of the slopes is -1 calculus will tell you how to compute these slopes how to
 compute
 the derivatives to check that these two curves $x^2 - y^2 = a$ and $2xy = b$
 to a and $2xy = b$
 equal to b are orthogonal to each other
 so that gives you a very nice example of an orthogonal
 trajectories two orthogonal trajectories here's a picture the red curves are
 the

curves $2xy = b$ and the blue curves are $x^2 - y^2 = a$ already

in the picture that you see that at the point of intersection they appear to be orthogonal the

picture is fairly accurate thanks to mathematica and here i would like to express my

thanks to one of the students of the mathematics department aditya maheshwari who generated this picture and the next one okay let us take another example from complex analysis

let us take a function from which is defined on $\mathbb{C} \setminus \{0\}$.

what is this function f of z equal

to $z + 1/z$ this function is defined on the whole of complex plane it except at 0

when it's not defined let us see what happens when z traces a circle $r \cos t + i r \sin t$

so z in

the domain traces a circle $r \cos t + i r \sin t$ remember that we are thinking of the complex

num numbers as points in the argan plane and you're familiar with this geometrical

representation of complex numbers

so i can think of the complex number z as $r \cos t + i r \sin t$ and let t vary

so you get a circle the z traces a circle in the domain what happens to f of z what is f of $r \cos t + i r \sin t$ what happens to the image curve let's see it's an

amusing exercise to understand these image curves

so what is f of z if z is $r \cos t + i r \sin t$

then what is f of $z = r \cos t + i r \sin t + 1/(r \cos t + i r \sin t)$ you have to of course use the complex conjugate or the denominator

and rewrite the thing and you will get $r + 1/(r \cos t + i r \sin t)$ upon $r \sin t$

t that is displayed as $1.$

42 in the slide

so as t varies what does this $1.$

42 represent $1.$

42

is the parameterization of an ellipse $1.$

42 represents an ellipse as you

very well know a $\cos t + i b \sin t$ is an ellipse okay what are the major axis

of the ellipse the semi-major axis is $r + 1/r$ what is the semi-minor axis $r - 1/r$

$1/r$

so it is an ellipse with semi-major axis $r + 1/r$ and semi-minor axis $r - 1/r$

r what is the relationship between the semi-major axis the semi-minor axis and the eccentricity b

squared equals $a^2 - b^2$ okay

so you can compute the eccentricity of the ellipse what are the foci of this

ellipse

so what are the foci of an ellipse it is $a - e$ and $a + e$

zero

but what is ae remember b^2 equal to $a^2 - e^2$
so $a^2 - e^2$
 b^2 is $a^2 - b^2$ but what is $a^2 - b^2$ what is b^2
minus 1 upon r
so what is $a^2 - b^2$ it is 4 .
so $a^2 - e^2$ is 4
so $a - e$ is 2 .

so the foci

of the ellipse are $(2, 0)$ and $(-2, 0)$.

this is very interesting because no matter what

the value of r is you get the same foci all these ellipses all these image
curves as r varies you

get $r \cos t$ plus $i r \sin t$ that you get a whole bunch of concentric circles

the image curves are

all ellipses but all these ellipses have the same foci they are called

confocal ellipses they

all have the same foci

so this function $1/z$.

$f(z) = z + 1/z$ is called the
Zukowski function

so the Zukowski function takes circles which center at the origin to
ellipse if you take that z traces a circle with center at the origin the image
 $f(z)$

traces an ellipse but the foci of the ellipse are $(2, 0)$ and $(-2, 0)$

so these

are called a Zukowski ellipses

so the next slide is an exercise which I just

done for you determine the foci of the Zukowski ellipses what are they $(2, 0)$
and $(-2, 0)$

0.

for one value of r you don't get an ellipse for one very special value of r
which

one I'll leave it to you to figure it out next a similar problem instead of
taking z as a circle as tracing a circle let us assume that the z traces
ray

from the origin to infinity take ray the ray making an angle

θ with a positive x -axis

so how do you think about this ray t

$z = t \cos \theta + i t \sin \theta$ it is $z = t (\cos \theta + i \sin \theta)$

t varies from 0 to infinity and θ is fixed

so I want to find the image $f(z)$

so $f(z)$

is $z + 1/z$ and as t varies and θ is fixed what you get as the
image curves the image

curves are hyperbolas I am going to call them the Zukowski hyperbolas find the
foci of these

Zukowski hyperbolas no prizes for guessing they are going to be $(2, 0)$ and
minus $(-2, 0)$

0.

the hyperbolas are also confocal and they have the same foci as the ellipses
so the entire

family of conics is a confocal family of conics show that the zukowski
 hyperbolas cut the
 zukowski ellipses orthogonally again we have found an orthogonal system of uh
 curves
 you have found two families of curves a family of ellipses and a family
 of hyperbolas every ellipse will cut every hyperbola orthogonally we
 have a pair of orthogonal trajectories why is this interesting or why is this
 important
 it's interesting because the geometry is beautiful it's important because
 zukowski employed
 this in the construction of airfoils
 so there are applications to aerospace
 engineering and i given you a certain link for the website where you can find
 an article
 by nasa the by the glenn research laboratory where you will find details of
 how zikowski
 employed this in the construction of airfoils now the zukowski function can also
 be applied
 in a very different field other than aerospace engineering and that's what is
 going to be
 in the next slide but before i come to that i am going to look at one more
 exercise again
 from complex function theory here we take a different function a very nice
 function f from
 c to c is given by f of z equal to z squared show that the image of horizontal
 lines t plus ic
 so fix your c and let t vary
 so what does t plus ic trace t plus ic is a horizontal line passing
 through the point 0 comma c
 so if i take z equal to t plus ic what is going to be z squared it is
 going to be t squared minus c squared plus $2i$ dc
 so what is this trace out when z varies from
 minus infinity to plus infinity what happens what is the curve t squared minus
 c squared comma $2tc$
 they are parabolas find the focus of this parabola and does this focus depend
 on c you expect
 that the answer will be no and that will be a confocal family of parabolas
 but please investigate now we are taking the image of horizontal lines
 let us look at the image of vertical lines a plus it when t runs over the
 real numbers a
 a plus it is a vertical line passing through the point a comma 0 .

 so if i take z equal to a plus
 it what is z squared a squared minus t squared plus $2i$ a t
 so as a point in the argument plane
 we are looking at a squared minus t squared comma $2at$ this is a parameterization
 for a parabola
 there is a parametric form for a parabola again find this parabola find the
 focus of this parabola
 and examine whether this parabola depends upon a the next exercise show that
 the parabolas
 of exercise 19 cut the parabolas of exercise 20 orthogonally again we see a
 pair of orthogonal
 trajectories

so now we have obtained three examples from complex function theory we are not used any deep ideas from complex function theory the theory of functions of a complex variable is a deep branch of mathematics but we are not using any of that we are only looking at those things that are well within your syllabus namely how to square a complex number everybody knows how to do that how to compute $z + 1$ upon z so and just by using these elementary ideas of complex multiplication and complex division we have obtained three beautiful examples of orthogonal trajectories okay so now let us look at the picture again aditya has generated these pictures using mathematica you see the red parabolas and the blue parabolas you can see them intersecting orthogonally each blue curve intersects each red curve and at the point of intersection the angle of intersection is 90° degrees so please try this out yourself it's very amusing to do that now comes the promised application of zukowski function the zukowski function in the service of astronomy celestial mechanics and here we come to a very beautiful theorem called bolin's theorem which goes back to 1911. it is remarkable that the zukowski function finds applications in celestial mechanics and those zukowski ellipses and hyperbolas will feature in the proof of bolin's theorem it's an old result due to bolin that the map f of w equal to w^2 transforms the equations of planetary motion into this equation $d^2 w$ by $d s^2$ plus w equal to 0 . this differential equation is surely familiar to you by now of course that s is not the real time it is a rescaled time the zukowski map and the zukowski ellipses feature in the proof of bolin's theorem bolin's theorem says that there is a very explicit transformation which transforms a very complicated system of differential equations namely the differential equations that govern the motion of the planets around the sun the two body problem the sun and the planet just these two bodies the system of differential equation governing the motion of a planet around the sun can be transformed into this innocent looking differential equation by remarkable transformation the motion of the planets and the evolution of this differential equation are equivalent actually goes back to newton but the analytical form in which we are given is due to bolin you can easily use this idea to prove kepler's first law of planetary motion one can prove that the planets revolve around the sun in elliptical orbits with the sun at one of the foci reason is very simple you can easily

solve this
 equation $d^2 w$ by ds squared plus w equal to zero
 so we have to solve this differential equation
 $d^2 w$ by ds squared plus w equal to 0.
 now you see w is a complex number it has a real
 part x and it has an imaginary part y
 so this equation is really two equations $d^2 x$ by
 ds squared plus x equal to 0 $d^2 y$ by ds squared plus y equal to 0 there are
 two equations
 of simple harmonic motions you could take x to be \cos of s y to be \sin of s
 more generally
 you could take x to be $a \cos$ of s and y to be $b \sin$ of s and what do we get we
 need to get w
 of s equals $a \cos$ of s plus $i b \sin$ of s whereby what is the original
 trajectory remember
 that the function is f of w equal to w squared the original coordinate was z z
 was the moving point
 coordinates of the moving point of the planet
 so z will be w squared
 so the position of the planet
 z of s is w of s squared which is a squared \cos squared s minus b squared \sin
 squared s plus
 $2 a b i \sin$ causes
 so now if you take the real part of z and imaginary part of z
 and eliminate the signs and the cosines then what do we see we see that this
 is the
 parametric equation of an ellipse
 so for example call this real part as x of s call the imaginary
 part as y of s and
 so what is this equation of the ellipse eliminate the cosine of the sine
 first write \cos squared s in terms of \cos squared s \sin squared s in terms of
 \cos squared $2s$ \sin
 causes in terms of \sin $2s$ use the equation \sin squared $2s$ plus \cos squared
 $2s$ equal to 1 and you
 get this equation and the equation that you see $2x$ minus a squared minus b
 squared divided
 by a squared plus b squared the whole squared plus y squared by a squared b
 squared
 equal to 1 this is an ellipse foci one of whose foci is at the origin it's a
 very
 easy exercise in coordinate geometry and i urge you to do it
 so what have we proved here we used
 bolin's theorem to prove that if the transformed equation of w the w equation
 is an ellipse
 the z equation is a square of the w equation and the square of the ellipse is
 another ellipse
 whose focus is at the origin
 so the z the locus of z is a ellipse with one focus at the origin
 namely the kepler's first law has been established okay although bolin's
 theorem is a theorem on
 differential equations we shall not prove it here because it is really not in
 the syllabus but
 i just want to tell you that the theory of differential equations goes goes
 out and reaches

out to various parts of physical sciences physics astronomy cartography
aerospace
engineering and this theorem was already known to newton and it goes under the
name of dual laws
of attraction i won't elaborate on what these dual laws of attractions mean
the analytical form in
which we have stated the theorem is due to bolin in the following two slides i
am going to give you
two references and these two references are books written by great masters one
of them is a great
mathematician and the other is a great physicist first one is v i arnold he
wrote a book
titled huygens barrow newton and hulk and we began this series of lectures by
talking
about the achievement of newton that the work of newton transformed astronomy
from an empirical
science into a dynamical science this little book is a delightful journey
through times from
isaac barrow to modern discoveries in astronomy spanning two centuries of
post-newtonian era two
centuries following newton's principia the journey progresses with remarkable
ease through the
discovery of the kirkwood gaps in the rings of saturn with commentaries on
some of the
deep propositions in newton's principia on algebraic geometry bowling's
theorem appears
in the appendix age 94.
the discussion is persuasively geometric and elegant the
triumphs of celestial mechanics on page 69 the notorious three-body problem
the monumental
works of pier simon laplace and finally poincare poincare is revolutionary
new
methods in celestial mechanics are expounded by one of the most significant
contributors to modern celestial mechanics the second reference is by the
nobel
laureate s chandra sheker he wrote a book newton's principia for the common
reader it
was published by oxford university press 1996.
this work of great scholarship is rich in content
and despite the title it is no easy reading but it's well worth the effort the
common
reader doesn't mean the common man it means you the students who are
concurrently learning
calculus they are learning physics they learn learning chemistry they are
equipped with
enough knowledge of physics and mathematics it is for such students that this
book has been
written of course this book is a 600 page book and not to be read from page 1
to page 600
linearly i recommend you directly go to page 57 and you see complete proofs of
the three
laws of planetary motion by kepler using the newton's laws of motion and its
universal
law of gravitation after this you could move on to reading the supplement to

chapter six where he discusses the dual laws of planetary attraction and you will see a proof of bolin's theorem there also and then you can move on to chapter seven on the kepler equation and the theorem on algebraic geometry which is contained in newton's principia i cannot resist quoting a charming essay of s chandrashekar and the essay compares the genius of newton with that of michelangelo the reference is given as item number three in this slide chandrashekar compares newton's writings of the principia with michelangelo's painting of the ceiling of the sistine chapel in vatican city chandrashekar considers these as works of the most rarified level of creativity let us see what chandrashekar has to say i will just give you one excerpt from this beautiful essay and i strongly recommend you to read this essay for its lovely style as well as a content the principia and the frescoes both in the realms are supreme unsurpassed expressions of human creativity i don't want to say more about these but i hope it this has enticed you into reading this essay of gender shaker on newton and michelangelo now let us come to differential equations in orthogonal trajectories suppose we are given a one parameter family of curves c_1 is it always possible to find an orthogonal system of trajectories c_2 we have seen many examples of pairs of orthogonal trajectories but now i give you a one parameter family of curves and i ask you whether there is an orthogonal trajectory and how to find it to understand this problem let us assume that the system of curves c_1 gives rise to a differential equation $m dx + n dy = 0$. we have seen examples you've got a system of curves with a parameter c you differentiate you get one more equation you eliminate c between the two equations and you get the differential equation we have seen numerous examples of that kind so now i assume that this has been done and you got the differential equation $m dx + n dy = 0$. now i already told you in the last lecture that this differential equation 1.43 is the differential equation for the phase curves of a first order system what is this first order system whose phase curves are equation 1.43 are given by 1.43 $\frac{dx}{dt} = n(x, y)$ $\frac{dy}{dt} = -m(x, y)$ the vector $(n(x, y), -m(x, y))$ is tangent to the curves c and since orthogonal trajectories are orthogonal to the original one the tangent

to the first becomes a normal to the second
so the tangent vectors to the orthogonal
trajectories must be $m\mathbf{i} + n\mathbf{j}$ where the tangents to the two systems are
perpendicular
to each other

so tangent to the first will be the normal to the other
so the orthogonal trajectories
are phase curves of this system $\frac{dx}{dt} = m$ and $\frac{dy}{dt} = n$
so the differential
equation for the orthogonal trajectories is $n dx - m dy = 0$.

so given a system of
curves c_1 we obtain the differential equation and then we obtain the
orthogonal trajectories by
others differential equation

so first we are given c_1 from c_1 we obtain the differential equation
for c_1 and then we obtain the differential equation for c_2 solve this
differential
equation and get the family c_2

so that's theorem 1 whatever i said just now can be
summarized as this theorem if a one parameter family of curves is given by the
differential

equation $m dx + n dy = 0$ then the differential equation for the
orthogonal

trajectories is $n dx - m dy = 0$ i

so under broad conditions any one parameter
family of curves admits an orthogonal trajectory i say broad conditions
because we are not

really discussed existence theorems for differential equations and we are not
going to

and these existence theorems will be valid under suitable conditions but in
practical situations

these conditions will always be satisfied

so now let us try out this on some
specific systems of orthogonal trajectories that we encountered the first of
course

are the family of concentric circles what the differential equation for that
we

have derived it $x dx + y dy = 0$

so what are the differential equations for the
orthogonal trajectories $y dx - x dy = 0$ it's a variable separable
equation and

the solutions are $y = mx$ or $x = my$

so the orthogonal trajectories are lines

through the origin and everybody knows that a circle through which center at
the origin

cuts a line through the origin orthogonally let us take the next one let us
take the

hyperbolas $x^2 - y^2 = c$ differentiate the c straight
away disappears

you get $x dx - y dy = 0$ what the differential equation for the
orthogonal

trajectories $y dx + x dy = 0$ solve this differential equation

which is a variable separable equation and you get $\ln |xy| = c$ remove

the absolute values and you get rectangular hyperbolas cutting the members $x^2 - y^2 = c$ orthogonally now let us take another example you take this one parameter family of curves $y = c e^{-x^2}$ find the orthogonal trajectories you differentiate it $\frac{dy}{dx} = -2xy$ so the differential equation for the family of curves is $\frac{dy}{dx} = -2xy$ so we get $\frac{dy}{y} = -2x dx$ integrate to get $\ln y = -x^2 + \ln c$ so the differential equation for the family of curves is $\frac{dy}{dx} = -2xy$ so the differential equation for the orthogonal trajectories is $\frac{dy}{dx} = \frac{1}{2y}$ we shall regard x as a function of y and integrate the solutions are $x = \sqrt{ay}$ where a is a positive constant remove the absolute value and you get a family of orthogonal trajectories note that $x = 0$ is not included in here because we are dividing by x when you do the method of separation of variables and $x = 0$ is also a special solution and we get that is understandable because all the curves of the original system are bell shaped curves they all have horizontal tangents when they cut the y axis and so the removal of the absolute value term will give you $x = \sqrt{cy}$ where c is any constant well now we come to a couple of examples so determine the orthogonal trajectories for the family of parabolas $y = kx^2$ that's the first problem the second problem is consider the family of circles touching the y axis at the origin find the differential equation for the orthogonal trajectories mind you i'm only asking you to find the differential equations for the orthogonal trajectories integrating the differential equations will be done in the next chapter when you discuss homogeneous differential equations and then consider the function f of z equal to $1/z$ upon z find the images of horizontal and vertical lines under this map f of z equal to $1/z$ upon z obtain the family of curves and relate related to exercise 23. we have done a similar exercise with parabolas now i am asking you to do the same exercise with f of z equal to $1/z$ upon z you will get a fourth example coming from complex analysis and the last slide is about two more exercises for today's lecture find the differential equation for the orthogonal trajectories for the one parameter family of coaxial circles one point 1. 37 remember that s

$1 + \lambda s^2 = 0$ but the circles had centers $(2, 0)$ and $(-2, 0)$ and radius 1 and we got a family of coaxial circles find the differential equation for the orthogonal trajectory i'm not asking you to solve the differential equation just obtain the differential equation and finally the family of confocal conics i asked you to determine the differential equation for this family of confocal conics and find out the differential equation for the orthogonal trajectories you will see something amazing happens and i'd like you to find an explanation for this phenomenon i think we'll stop here for today and we'll continue this series of lectures and in the next lecture i will discuss homogeneous differential equations thank you