

hello students welcome to the third lecture  
 in this series of differential equations we'll now move on to a slightly  
 different uh  
 perspective the geometric viewpoint we'll have a relook at the predator prey  
 model  
 so we take  
 the volterra load car system of differential equations that we saw on the very  
 first lecture  
 it says  $\frac{dx}{dt} = a - bx$  and  $\frac{dy}{dt} = ky - cx$   
 as you see in the  
 slide that is just displayed what does it mean to say that we have a solution  
 of this differential  
 equation we have to find  $x$  as a function of time and  $y$  as a function of time  
 put these two  
 functions of time together and form the pair  $(x(t), y(t))$   
 so now this is now a point  
 moving in the plane the pair  $(x(t), y(t))$  is a parameterized curve in the  
 plane one  
 would like to know what this curve looks like once again solving the system of  
 differential  
 equations means finding two functions  $x(t)$  and  $y(t)$  and  
 so we get a parameterized curve  $(x(t), y(t))$   
 $y(t)$  let us call this curve  $C$  in the  $xy$  plane now we want to understand the  
 cartesian  
 equation of this curve how do we find the cartesian equation of the curve we  
 are given  $\frac{dx}{dt}$  and we are given  $\frac{dy}{dt}$   
 so let us apply the chain rule and let us write  $\frac{dx}{dy}$  is  $\frac{dx}{dt}$   
 divided by  $\frac{dy}{dt}$  the dot over the  $x$  represents the time derivative  
 so  $\frac{dx}{dy}$  is  
 going to be  $x \dot{y}$  but what is  $x \dot{y}$  look at the first equation here  
 $a - bx +$   
 $ky - cx$  what is  $y \dot{x}$  look at the second equation  $ky - cx$  the  $x$  factors  
 from the  
 numerator and the  $y$  factors from the denominator and  
 so we get this equation 1.  
 14  
 which is obviously a variable separable equation and you know by now how to  
 deal with this  
 variable separable equation 1.  
 14 all right  
 so let us look at this equation 1.  
 14  $x$  and  
 $y$  denote populations and the population has to be non-negative it has to be  
 positive  
 so we're  
 looking at the differential equation in the first quadrant okay  
 so we may assume without loss  
 of generality that  $x$  and  $y$  are both positive also let us assume that the  
 denominator is not  
 zero the numerator is not zero that is  $b - c$  is not zero and  $k - c$  is  
 not zero  
 so let  
 us steer away from these two uh points and so what do we get we separate the

variables and we

get we separate the variables here we divide by  $x$  and multiply by  $k \ln x - c x$  and

so we get

this  $k \ln x - c x$  upon  $x^d$  by  $dy$  equal to  $b y - a$  upon  $y$  well integrate both

sides with respect to  $y$  and what do you get you get  $k \ln x$  there's no need to put the

absolute value because  $x$  is positive similarly  $y$  is positive

so there will be no modulus sign

under the logarithm

so we get  $k \ln x - c x + a \ln y - b y$  equals a constant the constant of integration has been denoted by the letter capital  $e$   $y$  capital  $e$  we shall see in a

moment note that 1.

15 is the cartesian equation of the parameterized curve  $x(t), y(t)$

so you are

familiar with this take a circle in the plane  $x = \cos \theta$   $y = \sin \theta$

those are the parametric equations of a circle or the cartesian equation  $x^2 +$

$y^2 = 1$  or you could take  $x = a \cos \theta$   $y = b \sin \theta$

the parametric equations of an ellipse the cartesian equation is  $x^2/a^2 +$

$y^2/b^2 = 1$ .

to give a third example from coordinate geometry we

could be looking at the parabola  $y^2 = 4ax$  that is a cartesian equation

the parametric equation is  $x = at^2$  and  $y = 2at$ .

so you know you know

this transition between parametric equations and cartesian equations

so what you see is that

equation 1.

15 over here is the cartesian equation for the curve  $c$  for this curve  $c$  namely the

curve the solution of this differential equation gives rise to a curve  $x(t), y(t)$  this curve

is denoted by letter  $c$  and what we have found is the cartesian equation this equation

1.

15 curve itself is called the phase curve the curve itself is called the phase curve for

the system of differential equations all right note that this equation 1.

15 is a very interesting

equation it's a very interesting equation what is 1.

15 say it says that  $k \ln x - c x + a$

$\ln y - b y$  is always constant in other words it means that this combination that you

see  $k \ln x - c x + a \ln y - b y$  this combination is always constant

its value is  $e$  it doesn't change with time which means that this is going to be some kind of a conserved quantity this combination is constant in time it

does not vary in time now if you have a mechanical system a system coming from classical mechanics a conservative system then you know that the energy is going to be conserved if you take a pendulum equation or the harmonic oscillator simple harmonic oscillator you know that the total energy is always conserved the conservation of this quantity is very similar to the conservation of energy and let us simply call it the conservation of ecological energy in some sense well whatever just by analogy i am just calling it conservation of ecological energy that's why i use the letter  $e$  to denote this constant of integration now let me say a few more things about phase curve it is also natural to ask for a picture of these phase curve how does this equation if you take this 1.15 you take equation 1.15 and you ask for a plot of this curve in the  $x$   $y$  plane how would these curves look like this equation is a complicated equation had it been  $x^2 + y^2 = e$  then it's easy for you to plot them they are concentric circles if  $e$  becomes large the radius becomes large you get a larger circle so as  $e$  keeps increasing you get a concentric circles but unfortunately this is not the equation of a circle it's a more complicated equation how does one sketch this curve it is not difficult to sketch this curve but what you would need in order to do so will be some basic concepts from calculus of several variables you will need a few concepts from the calculus of several variables and that would take us a little outside the scope of the present course and so i shall not get into the nitty-gritty of how to sketch those curves i will just make one comment that these curves are closed curves in the plane just as a family of circles or the family of ellipses are closed curves this family 1.15 as  $e$  varies they are family of closed curves if you keep changing  $e$  if you make  $e$  larger and larger the curves the closed curves will become larger and larger to understand how to draw these curves i'll just give you a reference for those whose curiosity has been aroused you can consult this book by paul glendening the title of the book is stability instability and chaos it's a very interesting book and those of you who want to understand how to sketch these curves for the volterra lotka model may consult this book i like also point out a very nice uh article in the in the website

which you can download it's available for free and these notes contain a very beautiful picture of these phase curves the family of curves given by 1.

15 has been

plotted in this article you could try that okay happy reading

so now let us look at the phase diagrams that arise in physics

so let us look at a simple looking differential equation  $\frac{dx}{dt}$  equal to  $y$   $\frac{dy}{dt}$  equals minus  $x$  equation 1.

16 equation 1.

16 is really the simple harmonic

motion the equations of the simple harmonic motion can also be written like like this

with the frequency  $\omega$  squared equals 1.

now i want to understand the phase curves for

1.

16 now you can directly look at 1.

16 and you can see that  $x$  equal to  $\sin t$   $y$  equal to  $\cos t$  is a solution of this equation 1.

16

so what do you think are the phase curves for 1.

16

they are very simple they are circles

so the phase curves of 1.

16 are the curves are  $\sin t$  comma

$r \cos t$   $x$  of  $t$  equal to  $r \sin t$   $y$  of  $t$  equals  $r \cos t$   $r \sin t$  comma  $r \cos t$  they are circles

in the plane all of them centered at the origin but let us try to see it in a slightly different

way let us divide one by the other and let us write  $\frac{dy}{dx}$  is again what what is  $\frac{dy}{dx}$

$\frac{y \dot{y}}{x \dot{x}}$  divided by  $\frac{y \dot{y}}{x \dot{x}}$  is  $\frac{dy}{dx}$  divided by  $\frac{dx}{dt}$  that is minus  $x$  upon

$y$  again that's a variable separable equation it's a variable separable equation proceed along

the usual lines and you will get  $x^2$  plus  $y^2$  equal to  $c$  we have got our concentric circles

so the phase curves are concentric circles simple though the system is the one important

comment that i'd like to make regarding this system equation 1.

17 carries with it less

information than the original system 1.

16 what do you mean by that what does it mean to say that we saw 1.

16 we have to find  $x$  as a function of time and  $y$  as a function of time what does

it mean to say we are solving 1.

17 solving 1.

17 simply means finding a relationship between  $x$  and

$y$  namely  $x^2$  plus  $y^2$  equal to  $c$  so getting the equation  $x^2$  plus  $y^2$

equal to  $c$  is very different from saying  $x$  equal to  $r \cos t$  and  $y$  equal to  $r \sin t$  the

circle  $x^2$  plus  $y^2$  equal to  $c$  has many different parameterizations

and  $\sin t$  comma  
 $\cos t$  is only one of the many parameterizations another parameterization could  
be  $x = \frac{1-t^2}{1+t^2}$   
 $y = \frac{2t}{1+t^2}$  you  
must  
have encountered this in your coordinate geometry courses or calculus courses  
the  
circle  $x^2 + y^2 = 1$  can be parameterized as  $\cos t$   
comma  $\sin t$   
it can be parameterized as  $\sin t$  comma  $\cos t$  it can be parameterized as  $1$   
minus  $t^2$  by  $1$   
plus  $t^2$  comma  $2t$  by  $1$  plus  $t^2$  there are many many different ways  
of parameterizing  
the circle  $x^2 + y^2 = 1$ .

so giving a relation between  $x$  and  $y$   
is far less informative than explicitly solving this differential equation 1.  
16

so 1.

17

carries less information than 1.

16 let me look at one more system and i will come back to this  
comment later

so consider the pair of equation  $\frac{dy}{dt} = 2xy$   $\frac{dx}{dt} = 1 + x^2$   
 $x$  squared

you work in the first quadrant well i'm going to divide one by the other

so i'll have to worry

about  $y = 0$  or  $x = 0$  don't worry about it work in the first quadrant  $x$   
bigger than  $0$   $y$

bigger than  $0$  okay now you want to get a first order equations out of 1.

19 you want to understand

the phase curves of 1.

19 what is it again  $\frac{dy}{dx} = \frac{y \dot{x}}{x \dot{y}}$  what is  $y \dot{y}$

is the derivative of  $y$  with respect to time and that is given by  $2xy$  what is  
 $x \dot{x}$  the time

derivative of  $x$  that is  $1 + x^2$

so  $\frac{dy}{dx} = \frac{y \dot{x}}{x \dot{y}}$  is  $\frac{y \dot{x}}{x \dot{y}}$  which will be  $\frac{2xy}{1+x^2}$

plus  $x^2$  again you see a variable separable equation you've got a  
variable separable equation

you know exactly how to deal with it

so i'll leave it to you to complete the problem question in this  
example 1.

19 can you explicitly integrate 1.

19 can you find  $y$  as a function of  $t$  and  $x$  as a function

of  $t$  satisfying this equation 1.

19 you can you can actually do that because you can first solve the

second equation  $\frac{dx}{dt} = 1 + x^2$  get your  $x$  pull it in the  
first equation and get

your  $y$

so here again is a case where the two equations can be solved explicitly  
to get  $x$  as

a function of  $t$  and  $y$  as a function of  $t$  whereas what i am asking you is a  
phase curve that means

the relation between  $x$  and  $y$  one more example consider the pair of

differential equation

$\frac{dx}{dt} = 1 + y^2$   $\frac{dy}{dt} = 1 + x^2$  so

problem is to find the phase curves of 1.

21 by obtaining a differential equation of the form 1.

20 that is get  $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$  what is  $\frac{dy}{dx}$  in this case it is going to

be  $\frac{1+y^2}{1+x^2}$  again a variable separable equation and you can

integrate this variable separable equation and you can find the relation between  $x$  and  $y$  please

complete this exercise and find the phase curves sketch the phase curves for various

values of the integration constant so we change the integration constant you will get

a different curve

so you get a family of curves

so please do that it's very easy you're going

to get  $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$  but you can guess what

it is going to be where  $\tan^{-1} x = \tan^{-1} y + c$

$c$  which is a constant of integration and

so you'll have to apply  $\tan$  to both sides

you get  $x = \tan(\tan^{-1} y + c)$  and

so you proceed and you get a relation between

$x$  and  $y$  okay now we go to the next issue let us look at the system  $\frac{dx}{dt} = y$

and  $\frac{dy}{dt} = -x$  again you do  $\frac{dy}{dx} = \frac{-x}{y}$  again divide one by the

other what do you notice the fee disappears the fee completely disappears you get  $\frac{dy}{dx} = \frac{-x}{y}$

$x^2 + y^2 = c$  which means again a variable separable

equation again you get  $x^2 + y^2 = c$

so the remarkable feature is

that no matter what this fee is the phase curves of 1.

22 are all circles  $x^2 + y^2 = c$

regardless of what  $\phi$  is in particular if i take  $\phi = 1$  if

i take  $\phi = 1$  what does 1.

22 read  $\frac{dx}{dt} = y$   $\frac{dy}{dt} = -x$  you

know how to solve that  $\frac{dx}{dt} = y$   $\frac{dy}{dt} = -x$  which means  $x$  of  $t$

equal to  $r \sin t$  and  $y$  of  $t$  equals  $r \cos t$  i change the  $\phi$  and i take the  $\phi$  of  $x$   $y$

to be equal to  $x^2 + y^2 = c$  then what happens then the solution is no longer  $\sin t$  and  $\cos t$

t you can check solution is going to change phase curves are  $x^2 + y^2 = c$

c the phase curves don't change that i just said a little while ago but this  $x$  equal to  $c \cos t$

and  $y = c \sin t$  will no longer work if i take  $\phi$  of  $x$   $y$  equal

to  $x^2$   
 it will work this one will work if I take  $\phi$  to be 1 if I take  $\phi$  to be  $x^2$   
 squared this is  
 not going to work anymore  
 so when I take  $\phi$  of  $x^2$   $y$  equal to  $x^2$  in 1.  
 22 I want you to find  $x$   
 as a function of time and  $y$  is a function of time given some initial  
 conditions  $x$  of  
 $\frac{1}{\sqrt{2}}$  equal to 1 upon root 2 and  $y$  of 0 you will now start scratching your head  
 how am  
 I going to do this  $\frac{dx}{dt}$  equal to  $yx^2$   $\frac{dy}{dt}$  equal to minus  $x^3$   
 how am I going to  
 find  $x$  as a function of  $t$  and  $y$  as a function of  $t$  because the  $\frac{dx}{dt}$   
 equation involves a  $y$   $\frac{dy}{dt}$   
 $\frac{dy}{dt}$  involves  $x^3$   
 so there it's a coupled system of equation and I cannot individually  
 solve  
 one of them and substitute into the other as it happened in the previous  
 example  
 so what you see  
 is that we are gradually progressing to more and more complicated equations  
 where finding  $x(t)$  and  
 finding  $y(t)$  individually is going to be not easy but I say that this is not  
 difficult either in  
 this example how to do it here is a hint the hint says you know that the point  
 $(x(t), y(t))$  lies in the  
 circle that you know  
 so  $x(t)$  is going to be  $c \cos(\psi(t))$   $y(t)$  equal to  $c \sin(\psi(t))$  where  
 $\psi$  is a function of  $t$  all I am saying is that  $\psi(t)$  equal to  $t$  will not  
 work it will work only  
 when  $c$  is 1 but  $\phi$  is not 1  $c$  is  $x^2$   
 so this function  $\psi(t)$  is going to be more  
 complicated  
 so any point on the circle is  $\cos$  of something  $\sin$  of something any  
 point  
 in the circle is  $(\cos \theta, \sin \theta)$  and  
 so the  $\theta$  becomes a function of  $t$  or in  
 this case I use the notation  $\psi(t)$   
 so I said  $x(t)$  equal to  $c \cos(\psi(t))$   $y(t)$  equal to  $c \sin(\psi(t))$   
 $t$  and I have to determine  $\psi(t)$   $c$  has to be 1 why is  $c=1$  because look at this  
 initial condition  
 $x(0)$  equal to  $\frac{1}{\sqrt{2}}$   $y(0)$  equal to  $\frac{1}{\sqrt{2}}$  and  $x^2 + y^2$   
 is  
 $c^2$   
 so  $c$  must be 1  
 so  $c$  is 1 that's clear  
 so you have to obtain a differential equation  
 for  $\psi(t)$  how are you going to obtain a differential equation for  $\psi(t)$   
 substitute  $x(t)$   
 $t$  equal to  $\cos(\psi(t))$  into the equation 1.  
 22  $y(t)$  equal to  $\sin(\psi(t))$  into the equation  
 1.  
 22 and you will get  $\dot{\psi}$  equal to something you will get a differential  
 equation for a  $\psi$  and

that differential equation is a variable separable equation and we can actually integrate and we can get it

so you see finding  $x$  explicitly as a function of  $t$  and  $y$  explicitly as a function of  $t$

has become slightly more complicated the exercise shows that when we change this function  $\phi$  what

happens the phase curves don't change the phase curves are always  $x^2 + y^2 = c$

what changes is the parameterization in the in one of the cases it is  $\sin t$  comma

$\cos t$  in the other case it is  $\sin \psi t$   $\cos \psi t$  and a  $\psi t$  is a slightly more

complicated function

so the parameterization has changed the curve is again the circle like i said

the circle has infinitely many parameterizations which is the correct parameterization that will

be the solution of the differential equation

so let me make a general remark

so we have seen a number of examples 1.

16 1.

19 and 1.

22 what is 1.

16 it is a harmonic oscillator  $\frac{dx}{dt} = y$   $\frac{dy}{dt} = -x$

16 is  $\frac{dx}{dt} = y$   $\frac{dy}{dt} = -x$  what is 1.

19  $\frac{d^2x}{dt^2} = -x$

22  $\frac{dx}{dt} = 1 + x^2$  again i repeat you can solve the second equation

obtain  $x$  as a function of time put into the first equation and then obtain  $y$  as a function of

time

so again here we are able to come do it and equation to 1.

22 which we have just now discussed

so in these three examples we are explicitly able to find  $x$  and  $y$  as functions

of time and this is a very rare situation it's a very rare situation usually it will not

happen take the volterra load equation  $\frac{dx}{dt} = -ax + bxy$   $\frac{dy}{dt} = cy - cxy$

you get this conservation of ecological energy this is all you can get you cannot go any further

and obtain  $x$  explicitly as a function of time and  $y$  explicitly as a function of time

so you

see there are limitations in this equation you have to be content with equation 1.

25

you cannot unlike the previous three examples obtain  $x$  and  $y$  individually as functions of

time but in practice this equation this phase curve is adequate for all

practical purposes it

turns out that the phase curve is adequate enough we can get all our information about the behavior of the system from equation 1.

25 let's do one more example let's look at the pendulum equation from your last lectures the pendulum equation recall is  $d^2y/dt^2 + g/l \sin y = 0$  where  $y$  is the angular displacement from the mean position the pendulum is displaced

from the mean position and set into oscillation and the displacement from the mean position is  $y(t)$  the angular displacement

so what is the angular velocity the derivative of angular displacement is the angular velocity  $dy/dt = z(t)$  there is angular velocity

you differentiate the angular velocity you get the angular acceleration

so  $dz/dt$  is

$d^2y/dt^2$  the angular acceleration but the differential equation gives you that

$d^2y/dt^2 = -g/l \sin y$

so let us consolidate these things together  $dy/dt = z$  and  $dz/dt = -g/l \sin y$  now look at equation 1.

27 that's

exactly what I attempt  $dy/dt = z$   $dz/dt = -g/l \sin y$  again you see that we have

got a system of equations a coupled system of differential equations finding  $y$  explicitly as

a function of time and finding  $z$  explicitly as a function of time is going to be a difficult

business and put the two functions together  $y(t)$  comma  $z(t)$  and you get a parameterized curve and

these parameterized curves are the phase curves for the pendulum equation either 1.

26 or 1.

27

they are really the same

so the pair  $y(t)$  comma  $z(t)$  are the phase curves for the pendulum equation 1.

26

so obtain a relationship between  $z(t)$  and  $y(t)$  obtain a relationship between  $z$  and  $y$  in

other words obtain the cartesian equations for the phase curves obtain the cartesian equations

for the phase curves and interpret your result physically you may have seen the picture of this

phase diagram if you plot the phase curves for various values of the integration constant you

will get a one parameter family of curves in the  $y-z$  plane and this picture is called

the phase diagram you have probably seen this phase diagram for a pendulum equation in your

physics courses you also have probably seen the physical interpretation for this relation between

z and y there is a relation between z and y right and the relationship has a meaning what is it it is the law of conservation of energy the pendulum is a conservative mechanical system and the law of conservation of energy gives you relationship between z t and y t and that relationship is what you will get as a cartesian equation for the phase curves you can also use a physical intuition to sketch these phase curves these phase curves are going to be closed curves because when you set the pendulum in oscillation the pendulum starts swinging and it exhibits oscillatory motion and so these phase curves are going to be closed circuits however if you give the pendulum a sharp push if you push it too hard what the pendulum do it will go it will go on it will go right on the top and will complete the circuit and will come down and it will x and it will exhibit circular motions and that will also enable you to accordingly draw the phase curves for large values of the energy try that out or maybe consult some physics books whatever the purpose of this exercise is to bring out the connection between the phase curves and physics the law of conservation of energy finding individually y of t and z of t as functions of time is going to be difficult it is going to involve elliptic functions in the last lecture i stopped with the elliptic functions and here we encounter these again these elliptic functions appear very naturally in physics we get easily the law of conservation of energy which is the relationship between the uh functions y t and z okay now we come to the next phase of this series of lectures namely differential equations of the form  $m \frac{d^2 x}{dt^2} + n \frac{d^2 y}{dt^2} = 0$  in books you will often see a differential equation being written like like equation 1. 28 in the slides  $m x'' + n y'' = 0$ .

now this equation 1.

28 is somewhat controversial because what is the meaning of this what is the meaning of the dx and the dy floating around we have been constantly saying that d y by d x in calculus is not d y divided by d x it is the derivative of y with respect to x it is just a symbol d y by d x it is not d y divided by d x d y is not a number and d x is not a number so here in equation 1.

28 suddenly the d x and the d y has been have been separated they were inseparable right in in your differential calculus and integral calculus

courses now they are being separated oh how cruel well the expression  $m \frac{dx}{dt} + n \frac{dy}{dt}$  can be

defined in mathematics in very precise terms there is a way to do it but we shall not do so

here because this is not the place to do that

so what do we do

so what do we do with equation

1.

28 we must clarify the meaning of this equation 1.

28 before proceeding because numerous

differential equations are going to be presented in the form 1.

28

so it is urgent to actually

clarify the meaning of  $m \frac{dx}{dt} + n \frac{dy}{dt}$  and we shall do

so now i should now proceed to clarify

this

so first let us begin by recalling the discussion that we have had

so far we have seen

that differential equations that arise naturally in the physical sciences and the biological

sciences are usually systems of differential equations they're pairs of differential equations

volterra load chi equations  $\frac{dx}{dt} = -ax + xy$   $\frac{dy}{dt} = cy - xy$

equal to  $k y - c$

$x y$  the simple harmonic motion  $\frac{dx}{dt} = y$   $\frac{dy}{dt} = -x$

again a system the

pendulum equation that we have just seen can be written as a system

so we're looking

at systems of differential equations and if in the mechanical system will be looking at the time derivative

so what kind of differential equations have you encountered so

far they are the form 1.

29  $\frac{dx}{dt} = nxy$   $\frac{dy}{dt} = -mxy$  these are the kinds of differential equations that we have encountered so far

so what does this mean for a mechanical system

i just said  $t$  will represent the time variable and therefore  $n$  of  $x y$  and  $-m$  of  $x y$  are

the components of the velocity if  $x$  and  $y$  are components of the displacement then  $\frac{dx}{dt}$  comma

$\frac{dy}{dt}$  is the vector of velocity in other words  $n$  and  $-m$  are the components of the

velocity of the particle and the problem of solving this differential equation 1.

29 amounts

to finding  $x$  explicitly as a function of time and  $y$  explicitly as a function of time but we have

just seen that in practice it is rarely possible to obtain from 1.

29  $x$  as a function of time and  $y$

is a function of time the classic example is that of the voltage allot guy equations even when

you can do it it is not entirely easy to do it okay

so we have to be content with

simply the phase curves the phase curves are much easier to obtain as we have seen time and again divide one by the other  $\frac{dy}{dx} = \frac{y \dot{y}}{x \dot{x}}$

which is minus  
m upon n and 1.

30 is a differential equation connecting x and y but we could have equally well proceeded differently instead of dividing  $\dot{y}$  by  $\dot{x}$  we do it the other way round we get  $\frac{dx}{dt}$  divided by  $\frac{dy}{dt}$   $\dot{x}$  divided by  $\dot{y}$  or  $\frac{dx}{dy}$  equal to minus n upon m so

we got on the one hand the differential equation 1.

30 and we got the differential equation 1.

31 and when you go back to the system 1.

29 there is no distinction between x and y x and y are variables which enjoy equal status so there's no partiality there's no need to be favorable to x or to y so both of them receive the same importance so it's not clear to me whether 1.

30 or 1.

31 is preferred these two equations have equal importance so in

view of the symmetry between x and y in view are the equal roles played by x and y we denote by

1.

28 either equation 1.

30 or equation 1.

31

so 1.

28 is a way of combining both 1.

30 and 1.

31

and writing it as a single equation

so 1.

28 should be interpreted as either 1.

30 or one point three

one okay

so this basically clarifies the meaning of this expression  $m dx + n dy$  equal to zero

sure we have separated  $dx$  and  $dy$  we have been cruel  $dx$  and  $dy$  were inseparable but we

have separated them but the interpretation has been given a precise interpretation has been

given 1.

28 should be thought of as 1.

30 or 1.

31 so now we have given a very clear interpretation

of  $mdx + ndy$

so from a practical point of view interesting first order equations like 1.

28

1.

28 is an equation that arises from systems 1.

29

so the study of equations like 1.

29 gives rise

to differential equation 1.

28 and 1.

28 i repeat is simply 1.

30 or 1.

31 depending on whether

you want to give  $x$  a more favorable position or whether  $y$  has the more favorable

status okay

so this is stated differently the pair 1.

29 is more fundamental and that is

a more interesting object under consideration 1.

29 is the basic object of study

and 1.

28 is simply an auxiliary tool and unfortunately in practice it is not possible

to completely solve 1.

29 it is not possible to get  $x$  as a function of time and  $y$  as a function of time all we can do is to solve this equation 1.

28 that is why equation like  $m dx + n dy$

equal to 0 are taught in great detail because this is all that we can really solve all

right another point i want to make now let us look at a slightly different system we taken the system

we have taken the system  $\frac{dx}{dt} = n - y$   $\frac{dy}{dt} = -m - x$  what i do is that i

multiply the right hand side of both of them by the same factor  $\mu$  of  $x$   $y$  remember we did one

simple example  $\frac{dx}{dt} = y$   $\frac{dy}{dt} = -x$  where  $\phi$  of  $xy$  is any function of  $x$  and  $y$  what did we see we saw that the

phase curves don't change the phase curves are always the circles what changes then what changes

is when you change the function  $\phi$  the parameterization changes

so now we are looking at the more general

situation i do the same thing start out with this equation  $\frac{dx}{dt} = n - y$   $\frac{dy}{dt} = -m - x$

equal to

$\frac{dx}{dt} = n - y$  and multiply the right hand sides for the same factor  $\mu$   $\mu$  of  $x$   $y$  again you will

see divide one by the other the  $\mu$  disappears if the  $\mu$  disappears that means that introducing

the factor  $\mu$  will not change the phase curves so phase curves of 1.

32 and the phase curves of 1.

29

are identical what changes the parameterization will change why would the parameterization change

think in terms of physics what does equation 1.

29 tell you it tells you that the components of

the velocity are  $n$  and  $-m$  what have i done i have changed the velocity by putting in a factor

of  $\mu$  of  $x$   $y$  a scalar factor  $\mu$

so the velocity changes its magnitude but the direction doesn't change

so the the speed of the particle changes but the trajectory of the particle

doesn't  
change the phase curve doesn't change the velocity along the phase curves has  
changed so  
parameterization along the phase curve will change  
so the phase curves of 1.  
32 are also the same as the phase curves of 1.  
29 so  
it doesn't matter whether i convert 1.  
29 into 1.  
32 because we have agreed that the phase  
curves are the only things that we can ultimately determine because it is very  
rare to actually  
find  $x$  as a function of  $t$  and  $y$  as a function of  $t$  this is a very important fact  
which is usually not  
emphasized in books and i'm i'm going slow with this because this is a very  
important point now i  
want to draw your attention to a certain article by john carlo rota what we  
have done in the last  
couple of slides is that we have meticulously elucidated the meaning of this  
expression  $m dx + n dy$  equal to 0 you will wonder why is it that i am spending  
so much time on this matter and this  
is based on a philosophical article written by a great mathematician giancarlo  
rota who is no  
more and he has written his philosophical view about the teaching of  
differential equations  
and this article is available online and rota's comments about  $m dx + n dy$   
equal to 0 have prompted me to spend a long time on elucidating this meaning  
of this  
expression however i must hastily add a disclaimer the article is a long  
article  
and there are many points that rota addresses in this article and i am in  
agreement with some of  
them particularly the one that i just said but generally i am in disagreement  
with many  
other points addressed by rota  
so this is a intellectual disagreement because this is really  
philosophical issues and pedagogical issues it is not that the mathematics is  
incorrect  
the mathematics is absolutely precise what we are talking about is  
interpretations  
the philosophical underpinnings and the pedagogical issues what to emphasize  
in courses and what not to emphasize in courses and i i agree with rota on a  
couple of things and  
i disagree with him on many things but that's life okay  
so now let us look at some more  
geometrical aspects of differential equations i like geometry  
so i spend more time on geometry  
we have seen that a differential equation  $m dx + n dy$  equal to 0 gives  
you a family  
of curves what is that family of curves the phase curves of the system the  
phase curves  
as a system it's a one parameter family of curves like  $x^2 + y^2$

squared equal to

$c$  is a one parameter family of circles in the case of the world's in a lot chi equation

you will get a one parameter family of closed curves filling the first quadrant in the case of

the pendulum equation you're again going to get a one parameter family of curves and you're going to

look at some physics books for the phase diagram

so equation 1.

28 gives you a

one parameter family of curves conversely given a one parameter family of curves

can we obtain a differential equation out of it yes we can one parameter family of curves are very pretty objects they appear in electrostatics

they appear in fluid mechanics how do they arise in electrostatics think of the equipotential lines if equipotential surfaces you got a distribution of charges then there are

surfaces which are called equipotential surfaces take these equipotential surfaces and

slice them by the  $x$   $y$  plane and you get equipotential lines on the  $x$   $y$  plane for example

i strongly recommend you to turn to page 635 of resnick and halliday's book i already mentioned

this book earlier and on page number 635 or the same edition mind you there are some

beautiful pictures of equipotential lines now let us look at some interesting examples of such families the simplest example is a family of concentric

circles differentiate this equation you get  $2x$  plus  $2y$   $dy$  by  $dx$  equal to  $\theta$  the constant

$c$  disappears when you differentiate divide by 2 you get  $x$  plus  $y$   $d y$  by  $d x$  is  $\theta$

that's the differential equation

so 1.

33 is a differential equation for the one parameter family

of curves  $x$  squared plus  $y$  squared equal to  $c$  pretty easy but now let us proceed to

another example where it's not going to be

so user friendly let us consider the family

of circles touching the  $y$ -axis there is a  $y$ -axis draw a circle touching the  $y$ -axis at the origin

the center must be  $c$  comma  $\theta$  and the radius is again  $c$  and

so what is the equation of the

circle  $x$  minus  $c$  the whole squared plus  $y$  squared equal to  $c$  squared the  $c$  squared

cancel out and you get equation 1.

34 on the slide differentiate 1.

34 with respect to  $x$  and

divide by 2 you get  $x$  plus  $y$   $d y$  by  $dx$  equals  $c$

so from  $c$  put this value of  $c$  back into 1.

34

and you get a differential equation 1.

36 again use 1.

35 where we have the  $c$  and substitute it into 1.

34 and do a little bit of rearrangements you will get this beautiful differential equation

1.

36 and 1.

36 is the differential equation for this one parameter family of circles touching the y axis at the origin now i am going to give you a little exercise the exercise is not difficult but it is essential that you work it out we have regarded when we differentiate this equation 1.

34 we are assuming that y is a function of x that means y is implicitly a function of x is this a valid assumption sketch these circles and find out what happens

at the origin and find out what happens at  $2c$  comma  $0$ .

in the neighborhood of this point

is it legal to say y is a function of x should we not say x is a function of y should

we not regard x as an implicit function of y and should we not proceed by differentiating 1.

34 with respect to y instead of differentiating with respect to x this that's correct we must regard x as a function of y and we must proceed by

differentiating with respect to y but we get the same answer we again get get the same differential equation 1.

36 and that is something that you must check that's a little exercise for you it's

a very easy exercise and i urge you to do it now after you do this exercise do you see the

merits of writing equation 1.

30 and 1.

31 in the symmetrical form 1.

28 let me just go back few slides for you

so after you do this exercise do you agree with me that it is better to work with the form 1.

28 because sometimes you may have to regard y as a function of x and sometimes

you might have to regard x as a function of y as this example clearly shows and

so making the thing unsymmetrical is not advisable and working with a more symmetrical form of the equation

1.

28 is advisable it's a desirable feature all right

so you will agree with me that 1.

28 is the

most desirable feature now i'm going to give a little more exercises but this time it's going

to be fun with colored pens i want you to sketch those circles touching the y-axis at the origin

with blue pen next sketch circles in with red pen touching the x-axis at the origin and cutting

the blue circles at right angles

so do this sketching of these curves and obtain a beautiful picture your artwork in blue and red pens and then go back to resnicken holidays

book page 635 and look at figure 29.

15 does your picture closely approximate to this when the length of the electric dipole becomes smaller and smaller so with this interesting

exercise i shall close today's lecture thank you you

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