

hello

so welcome students to the second lecture of the series on differential equations let us briefly recall where we stopped we were looking at a very innocent looking differential equation  $dy/dt = y^2$  and we came up to this point and we found the solution  $y(t) = 1/(1 - ct)$  we observe that the solution as time  $t$  goes to  $1/c$  from the left  $y(t)$  goes off to infinity the solution blows up in finite time there's a finite time catastrophe as it were this is reason why this happens is because of this  $y^2$  term instead of  $y$  if i put  $y^3$  you will see the same thing happening in finite amount of time the solution would shoot off to infinity there is nothing wrong with the differential equation the differential equation is defined everywhere  $y^2$  is a polynomial nevertheless the solution does not live on the entire real line it lives from minus infinity to  $1/c$  that is the interval over which the solution exists is usually not the entire real line but only a part of the real line it's only a very small part of the real line usually so this is what i meant in the previous slide when i said that even though the differential equation will be defined everywhere the interval  $I$  that is  $I = (t_0 - \epsilon, t_0 + \epsilon)$  that interval  $I$  may not be the whole interval of time that is there may be escape to infinity in finite time the solutions may escape to infinity in finite time okay so let us now look at uh this uh phenomenon in a little more a little more detail maybe in couple more examples we can see before we take that up ah there is another small point that i'd like to address namely if you notice we have been using indefinite integrals i want to point out and i've been saying this to all my students that whenever possible work with definite integrals try to avoid indefinite intervals there are situations where you cannot do it or when in some situations it is probably not feasible or it's more clumsy to not to use it so try to use definite integrals whenever feasible that's the fundamental mantra i would like to convey why why should definite integrals be used because definite integrals are superior beings why are they superior beings notice that the integral  $\int_a^b f(x) dx$  is defined for any continuous function  $f$  over the interval  $[a, b]$  the symbol has a precise meaning whether or not you're able to compute the indefinite integral whether or not you're able to find the primitive for  $f$  of  $x$  this

symbol  $\int_a^b f(x) dx$  makes perfect sense another advantage is that when you use definite integrals in differential equations you automatically incorporate the initial conditions

so let us do the same problem again using different integrals instead of indefinite integrals

so again let us go back to the differential equation  $\frac{1}{y^2} \frac{dy}{dt} = 1$ .

now we shall integrate both sides with respect to  $t$  over a certain interval  $[0, s]$  that is we are looking at a definite integral

this time and what do we get  $\int_0^s \frac{1}{y^2} dt = \int_0^s 1 dt$

left hand side of course integrates to  $s$  what about the left hand side

remember look at the differential equation  $\frac{dy}{dt} = y^2$

so  $y'$  is positive right

so  $y$  is a strictly monotone increasing function and therefore i can you appeal to the substitution theorem i put  $y(t)$  equal to  $u$  and i get  $y' dt = du$  when  $t$  is  $0$  what was  $y$  of  $0$   $c$  and when  $t$  equal to  $s$  the value of the variable  $u$  will be  $y$  of  $s$

so the integral using the substitution  $y(t) = u$  transforms into  $\int_c^{y(s)} \frac{du}{u^2} = \int_0^s 1 dt$  now you can integrate  $du$  by  $u^2$  and then it will be  $1/u$  with a minus sign for the initial condition  $c$  and this  $c$  and  $y$  of  $s$  and you simplify and rearrange you get the same thing you get  $y$  of  $s$  equals  $c$  divided by  $1 - c/s$

so i'm omitting this little algebra this problem of integrating  $du$  by  $u^2$  putting in the limits doing the rearrangements and getting your  $y$  of  $s$  it's a very trivial exercise and i urge you to carry that out yourself and verify that finally you get  $y$  of  $s$  equal to  $c$  upon  $1 - c/s$  you recover the solution of this differential equation

but this time we have employed systematically definite integrals and not the indefinite integrals as in the previous slides okay and i once again like to remind you that definite integrals are superior because the definition makes perfect sense regardless of whether you are able to find the primitive or not whether you are able to find a function whose derivative is the given function the anti-derivative as it were integral of  $\cos x$  is  $\sin x + c$  because the derivative of  $\sin x$  is  $\cos x$  that is an anti-derivative of  $\cos x$  is  $\sin x$

in this case it is easy to figure out what the anti-derivative of  $\cos x$  is but in many cases as

you know determining that derivative is not easy it's tricky sometimes it's not possible at all for you to find the function whose derivative is a given function but when you deal with definite integrals definite integrals remember are defined as areas and they have a very precise and rigorous mathematical meaning so they are always superior objects all right so now let's go to two exercises consider the differential equation  $\frac{dy}{dt} = 1 + y^2$  squared initial conditions are prescribed  $y(0) = 1$ . does the solution escape to infinity in finite time what if the right hand side is replaced by  $1 + y$  to the power 10 so let us pause and think about how to do this question you see your initial impulse will be let us integrate this differential equation okay let's try it let's do it uh let's let's go whole hog and compute the integral and compute determine the solution and then answer this question whether the solution escapes to infinity in finite amount of time all right so let's see so your differential equation is  $\frac{dy}{dt} = 1 + y^2$  there is absolutely no problem in dividing by  $1 + y^2$  since it's always positive integrate over some interval say  $0$  to  $s$  what do you get when time  $t$  equal to  $0$  we have prescribed the initial condition to be  $1$   $\int \frac{dy}{1 + y^2} = t$  integral  $dt$  from  $0$  to  $s$  which is  $s$  that will give you  $\tan^{-1} y$  of  $s$  minus  $\tan^{-1} 1$  equals  $s$  or you get  $y$  of  $s$  equal to  $\tan(s + \frac{\pi}{4})$  so you got the solution well so from the solution you will say okay you will examine this solution you will examine the solution and then you will figure out when  $s$  approaches  $\frac{\pi}{4}$  from the left the solution  $y$  of  $s$  goes to plus infinity this is all very fine but let's go to the problem let's go back to the problem and see what's being asked what's being asked you are being asked does the solution escape to infinity in finite time you're not asked to determine the solution and the other problem is that the right hand side instead of  $1 + y^2$  would be  $1 + y^{10}$  now suppose you have  $1 + y^{10}$  and you try to solve the differential equation you will end up having to integrate  $\frac{dy}{1 + y^{10}}$  by  $1 + y^{10}$  and that's going to be exceedingly tedious it's going to be time consuming and tedious whereas all that is being asked is does the solution escape to

infinity in finite time all right well so question how on earth are you going to answer this question without explicitly finding the solution one can do that and let's see how to think about that it's very easy and it shows a very important principle in the theory of differential equations so what's given to you you're given  $\frac{dy}{dt} = 1 + y^2$  equals 1 plus y squared y of 0 equals 1. well you can certainly say  $\frac{dy}{dt}$  exceeds  $y^2$  surely you will agree with me that  $1 + y^2$  is bigger than  $y^2$  is strictly bigger as a matter of fact so now we can do the same thing now  $y^2$  is positive remember so  $1 + y^2 > y^2$   $\frac{dy}{dt}$  is strictly bigger than  $y^2$  now integrate now integrate over say  $0 < s < 1$  what do you get get integral when when  $t = 0$  when time  $t = 0$  what's the value of  $y$  value of  $y$  is 1 so it'll be  $\int_0^s \frac{1}{1 + u^2} du > \int_0^s \frac{1}{u^2} du$  greater than  $s$  when you integrate 1 you get  $s$  over the interval well now you can come continue the calculations and then so what do we do we do we do this minus  $\frac{1}{1 + u^2}$  ah from  $1$  to  $y$  of  $s$  greater than  $s$  that will be that will give me  $1 - \frac{1}{1 + y^2} > s$  bigger than  $s$  or  $1 - \frac{1}{1 + y^2} > s$  bigger than  $1 - \frac{1}{1 + y^2}$  so this will be this would mean  $y$  of  $s$  exceeds  $1 - \frac{1}{1 + y^2}$  upon  $1 - \frac{1}{1 + y^2}$  we immediately see that as  $s$  goes to 1 from the left the right hand side of the displayed inequality the displayed inequality in red goes to infinity does the solution  $y$  of  $s$  certainly cannot live beyond  $s$  equal to 1 in fact we have seen that the solution goes to infinity at  $\pi/4$  which is actually less than 1 now what have we done we have done a very simple thing we have gone from the differential equation to a differential inequality we simply knock off this 1 and we say  $\frac{dy}{dt}$  is bigger than  $y^2$  squared and the rest of the calculation was very simple so even if you have  $1 + y$  to the power 10 we can do the same thing we can simply knock off the one and you can say  $\frac{dy}{dt}$  exceeds  $y^{10}$  we can divide by  $y$  to the power 10 and proceed along the same lines nothing stops us from doing that so the point is that uh it's not necessary to solve the differential equation completely in is enough to get to replace it by something else and it's not necessary to solve the differential equation without solving the differential equation we can still draw our conclusions and that is the most important thing in the theory of differential equations one seldom solves the differential equation to completion one

often obtains behavior of the solutions without explicitly carrying out the solution to the end without determining the solution explicitly we try to obtain properties of the solution that's what the theory of differential equations is all about now let me insert an important comment note that we have passed from the differential equation  $\frac{dy}{dt} = 1 + y^2$  with  $y(0) = 1$  to the differential inequality  $\frac{dy}{dt} > y^2$  with  $y(0) = 1$  and then we concluded that  $y(s)$  is bigger than  $1 + s$  which means  $y(s)$  would already have gone to infinity by the time  $s$  goes to 1 note that we are not saying that  $y(s)$  becomes infinity exactly as  $s$  goes to 1 we are saying that  $y(s)$  goes to infinity either as  $s$  goes to one or perhaps before that that is the time of existence  $T$  of  $y$  cannot exceed one but it can actually be strictly smaller than one we actually saw this we actually integrated the differential equation the first display in the slide  $\frac{dy}{dt} = 1 + y^2$  with  $y(0) = 1$  we actually integrated this differential equation we saw we found that the solution is  $y(t) = \tan(t + \frac{\pi}{4})$  and we saw that the solution goes to infinity as  $t$  tends to  $\frac{3\pi}{4}$ .

so the solution becomes infinity before time  $t = 1$  so what our difference in inequality tells you is that the time of existence cannot be greater than 1 but it can be actually less than now the next problem is to answer the same question but instead of  $\frac{dy}{dt} = 1 + y^2$  i'm giving you  $\frac{dy}{dt} = 1 + y^2$  again you multiply by  $1 + y^2$  integrate both sides with respect to  $t$  use definite integrals recommended but you can also use indefinite integrals if you like that's your choice but here it's recommended wherever possible to use definitively so you can try uh the second problem yourself and try to figure out whether the solution escapes to infinity infinite amount of time or does it live forever all right so there are two simple exercises based on separating variables that is bringing the  $y^2$  on the left and integrating with respect to  $t$  and so on now let us take the third problem here we consider the problem  $\cos x \frac{dy}{dx} = 1 + \sin x$  again it's a variable separable equation remember what  $\frac{dy}{dx} = f(x)g(y)$  so this this factor  $\cos x$   $1 + \sin x$  you put it on the right hand side and what do you get the right hand side is

a function of two variables  $f$  of  $x$   $y$  and it's a product of a function of  $x$  times a function of  $y$   
 so whenever the right hand side that whenever you have  $dy$  by  $dx$  equals  $g(x)h(y)$  we refer to  
 the differential equation as a variable separable equation this is also a variable separable  
 equation the  $dy$  by  $dx$  is a product of a function of  $x$  alone times a function of  $y$  alone here of  
 course because of the presence of the cosine here under cosine there i am going to assume  
 that  $x$  and  $y$  lie in the open interval  $-\pi/2$  to  $\pi/2$  we are going to work in the open  
 interval  $-\pi/2$  to  $\pi/2$  okay all right the problem continues prove that the solution  
 $y$  of  $x$  is defined on the entire interval  $-\pi/2$   $\pi/2$  and as  $x$  approaches  $\pi/2$   
 the solution  $y$  of  $x$  approaches  $\pi/2$ .  
 discuss the special case with where the initial conditions are specified  $y$  of  $0$  equals  $0$ .  
 here for a change i'm giving you an indefinite integral just for a change let's work with indefinite integrals although they are  
 inferior beings never mind they also have a right to live as it were  
 so let us separate the variables as it were let us divide by  $\cos y$  and let us divide by  $\cos x$   
 so that the  $y$  variables are all on the left and the  $x$  variables are all on the right and perform an  
 integration you can integrate  $1/\cos y$   $1/\cos y$  is  $\secant y$  what are the integral of  $\secant y$   
 with respect to  $y$  it is  $\log \secant y + \tan y$  there is no need to put the absolute value because we  
 are in this interval  $-\pi/2$  to  $\pi/2$  where the thing is positive  $\secant y + \tan y$   
 $y$  what is that it is going to be  $1 + \sin y$  upon  $\cos y$  cosine is an even function it's  
 positive and  $1 + \sin y$  is also positive there is no need for an absolute value sine  
 so that is the integral of  $1/\cos y$  is  $\log \secant + \tan$  and then you have got  
 the integral of  $\sin y$  by  $\cos y$  that is integral of  $\tan y$   $dy$  and the integral is  $\log$   
 $\secant$   
 so you got sum of two logs it will be a lower product  
 so the left hand side will integrate as  $\log$  of  $\secant^2 y + \secant y$  table then you see a lot of symmetry  
 here what you see with  $y$  on the left you see the same thing with  $x$  on the right  
 so when you integrate the same song and dance it'll be  $\log \secant^2 x + \secant x$   $dx$   
 so you will get when you integrate this you will get a  $\log \secant^2 y + \secant y$  plus  $\secant x$  right time  
 let's just let me just slow down a little bit by actually in inserting a couple of lines all

right you agree on the left hand side you're getting  $\log \sec^2 y + \sec y \tan y$   
 $y$  equals  $\log \sec^2 x + \sec x \tan x$  plus a constant of integration  $c$   
 exponentiate  $\sec^2 y + \sec y \tan y$  equal to  $e$  to the power  $c$  into  
 $\sec^2 x + \sec x \tan x$   
 so here we stop and here we say that this is the solution well you might object you might say that the solution should be  $y$   
 equal to something that's not the form in which we got the solution what form have we got the  
 solution is described in implicit form you have studied implicit functions and implicit  
 differentiation in the differential calculus part of your curriculum you encounter implicit functions  
 and so here the  $y$  is a function of  $x$  but described implicitly this is going to happen in the  
 whenever you solve differential equations in the first order usually the solution will  
 present itself in implicit form as it happens here all right so the solution is given in  
 implicit form so let's go back to the slides yeah you see that  $\sec^2 y + \sec y \tan y$   
 $y$  equals  $e$  to the power  $c$  into  $\sec^2 x + \sec x \tan x$  what does the problem ask  
 you the problem look look at the problem again prove that the solution is defined on the  
 entire interval  $-\pi/2$  to  $\pi/2$  that's obvious that here the solution is defined  
 on the entire interval nothing happens when  $x$  on this interval i mean as long  
 as you're in the open interval there doesn't seem to be any kind of trouble in this equation  
 well let's look at what the problem is asking again as  $x$  tends to  $\pi/2$  you have to show that  
 the  $y$  of  $x$  also goes to  $\pi/2$ .  
 look at what happens on the right hand side  $\sec^2 x + \sec x \tan x$  what is that that is  $1 + \sin x$  upon  
 $\cos^2 x$  right hand side is  $1 + \sin x$  upon  $\cos^2 x$  as  $x$  approaches  $\pi/2$   $1 + \sin x$  approaches 2 and  
 the denominator  $\cos^2 x$  approaches 0.  
 and so this factor  $\sec^2 x + \sec x \tan x$  goes to plus infinity and that second factor is a constant  
 so compulsorily this  $\sec^2 y + \sec y \tan y$  must go to infinity incidentally notice that the  
 differential equation tells you that  $dy/dx$  is always positive remember  $\cos$  is positive  
 $\pi/2$  is positive on this interval there's no problem and

so  $\frac{dy}{dx}$  is positive  
so the solution

$y(x)$  is a monotone increasing function is a monotone increasing function and  
as  $x$  goes to  $\frac{\pi}{2}$  it is compulsory now that from the from this equation we  
compulsively see that  $y$  of  $x$  must also go to  $\frac{\pi}{2}$  beta my  $y$  of  $x$  monotone  
increases to  $\frac{\pi}{2}$

so that answers the question in the last example study what happens when  $x$   
tends to minus  
 $\frac{\pi}{2}$

so i am leaving it to you to study what happens to this function  $\sec^2 x$   
 $\sec^2 x$

plus  $\sec x \tan x$  what happens when  $x$  goes to minus  $\frac{\pi}{2}$  well what is it  
it is 1 plus

$\sin x$  upon  $\cos^2 x$  as  $x$  goes to minus  $\frac{\pi}{2}$  the numerator goes to 0  
and so

does the denominator

so it is a zero by zero form

so you know how to deal with such limits your

you're done many such limit problems and it's an amusing exercise for you to  
find out what happens

to the limit as  $x$  goes to minus  $\frac{\pi}{2}$  and so you likewise investigate what  
would happen to

the corresponding  $y(x)$  as  $x$  goes to minus  $\frac{\pi}{2}$  right

so that is something for you to think about

the next question is if the initial condition is  $y(0) = 0$  right if  $y$   
of 0 happens

to be 0 that is when  $x$  is 0 then  $y$  is also 0.

so what happens to the last displayed equation

that you see here in this slide in red you put  $x$  equal to 0 in this equation  
that

is displayed in red then what happens  $y(0) = 0$  remember

so  $x$  is 0 and  $y$  is 0.

the

$\tan x$  becomes 0 and the  $\tan y$  becomes 0 the  $\sec x$  becomes 1 and the  $\secant$   
 $y$  also

becomes 1

so what is left we get  $e$  to the power  $c$  equals 1

so the equation that is displayed

in red simplifies to  $\sec^2 y + \sec y \tan y = \sec^2 x + \sec x \tan x$

plus  $\sec x \tan x$   $e$  to the power  $c$  is what 1 and

so you simply get  $\sec^2 y + \sec y \tan y$

times  $y$  equal to  $\sec^2 x + \sec x \tan x$  this equation will force  
you that  $y$  equal to

$x$  or does it i think you should investigate that

so does it follow that the solution is given by

$y(x) = x$

so what happens to the equation  $e$  to the power  $c$  we figured out is 1 and

so our this equation reads  $\sec^2 y + \sec y \tan y = \sec^2 x + \sec x \tan x$

plus  $\sec x \tan x$

so from this does it follow that  $y(x) = x$  i think you should

definitely spend some moments and think about it we go to the next problem

solve the differential equation  $1 + e^t \frac{dy}{dt} + e^t - y = 0$  again the question is does the solution escape to infinity for some finite time solve the differential equation also does  $y(t)$  have a limit as  $t$  goes to minus infinity

so number of questions you are given a differential equation okay it's a variable separable equation you divide by  $e^t - y$  and you divide by  $1 + e^t$  so

we do that abbreviating  $\frac{dy}{dt}$  by  $y'$  we get  $e^t y' + e^t - y = 0$  you can immediately integrate this with respect to  $t$  and you get  $e^t y + \log(1 + e^t)$  is where  $c$  is a constant of integration the constant of integration almost obviously be positive the term  $e^t y$  in 1.

$12$  is positive and  $\log(1 + e^t)$  is also positive so the constant of integration must be positive okay

so the differential equation also shows so what does the differential equation tell you it tells you that  $y'$  is negative from the equation you immediately see that  $y'$  is negative the exponential functions are positive so from the equation you see that  $y'$  is everywhere negative so  $y$  is a monotone decreasing function why is the monotone decreasing function and suppose that the solution were to live on the whole interval zero to infinity suppose there is no escape to infinity in finite amount of time there are two possibilities two scenarios

the solution escapes to infinity there's a catastrophe or the solution lives forever suppose a solution lives forever then what could you do we could let  $t$  tend to infinity in 1.

$12$  let's go back to 1.

$12$

so this is an equation and if  $i$  could if this were to be valid for all values of  $t$  there is a solution were to live forever then  $i$  can allow the  $t$  to go to infinity if  $t$  goes to infinity what happens to the  $\log(1 + e^t)$  term  $\log(1 + e^t)$  goes to infinity the other term  $e^t - y$  is also positive so the left hand side goes to infinity whereas the right hand side is constant how is this possible we have a contradiction it is not possible that the solution lives on the whole interval  $0$  infinity the solution cannot live forever the

solution must escape to infinity in finite amount of time well will it go to infinity or will it go to minus infinity if  $y$  of  $t$  goes to plus infinity then  $e$  to the power  $y$  would go to plus infinity and 1.

12 for forbids this from happening

so in this case  $y$  of  $t$  must go to minus infinity and we should get  $e$  to the power  $y$  going to  $0$  what is the largest interval  $0 < t < \infty$  on which the solution is defined and what happens as  $t$  goes to that is what is the time of existence that is there is there is a finite time in which the thing is going to escape to minus infinity and that finite time is capital  $t$

so what is the value of capital  $t$

so little  $t$  goes to capital  $t$

so in the equation

1.

12 you will get  $\log$  of  $1 + e$  to the power capital  $t$  but as little  $t$  goes to capital  $t$   $y$  goes to minus infinity this  $e$  to the power  $y$  term disappears and again the right hand side is constant and

so you can calculate what the capital  $t$  is going to be what is the capital  $t$  it is going

to be given by  $e$  to the power capital  $t$  equal to  $e$  to the power  $c - 1$

so what is capital  $t$  is

going to be  $\log$  of  $e$  to the power  $c - 1$  so that basically answers this first question for you

i'll leave the second part for you to think over you you should have some food for thought we

will now look at the special case what happens when we look at this differential equation which

is variable separable  $\frac{dy}{dx} = g(x)h(y)$  into  $h(y)$  we remember we have been all the time

dividing by  $h$  we also been using the fact that  $g(x)$  is not zero well what is the problem with  $h$

vanishing if  $h$  becomes zero we can't divide by  $h$  if  $h$  is not zero we can still divide by  $h$  but what

about  $g$  being zero what is it what is the harm in  $g$  becoming zero the harm is that remember we have

been using the change of variables uh theorem the substitution theorem remember we have been using

the substitution theorem many times and the and we can only use the substitution theorem the

derivative is non-zero throughout the interval worry about what happens when this  $g$  of  $x$  is zero

okay again i say usually in real life situations when differential equations come to you they'll

come to you with prescribed initial conditions and

so the initial conditions are such that

why you have  $x$  naught equal to  $y$  naught where  $x$  naught is some given time and  $y$  naught

is the state of the system in time  $x$  equal to  $x$  naught now suppose if  $h$  of  $y$

naught happens to be zero supposing  $h$  of  $y$  naught happens to be zero in the right hand side then the right hand side is zero notice that the constant function satisfies the differential equation right when you differentiate a constant function the left hand side is  $0$  and the constant function  $y$  equal to  $y$  naught when you plug it in you see that the right hand side is also  $0$ .

so  $0$  equal to  $0$  yeah  
so the constant function  $y$  of  $x$  equal to  $y$  naught satisfies the differential equation it also satisfies the initial condition function is everywhere the  $y$  is everywhere equal to  $y$  naught and hence in particular at  $x$  naught also it is equal to  $y$  naught so we have solved the initial value problem namely the solution is the constant solution so the case when  $h$  is  $0$  can be handled in this very trivial fashion on the other hand if  $h$  of  $y$  naught is not  $0$  then we can divide by  $h$   $y$  remember we are only looking at the situation around in a small interval around  $x$  naught remember that the solution may not live for the whole interval of time and the entire drama is valid only in the neighborhood of the initial conditions so  $x$  is close to  $x$  naught and  $y$  is going to close to  $y$  naught  $h$  of  $y$  naught is not zero so  $h$  is not going to be zero near why not like continuity so i can divide by  $h$  of  $y$  naught but  $g$  of  $x$  naught could be zero in which case our use of the change of variables formula our use of the substitution theorem is suspect in which case what we shall do is that we shall assume that  $g$  is zero  $g$  is zero at say finitely many points in between places where  $g$  is  $0$   $g$  is going to be strictly positive or strictly negative and we have to analyze the situation on the different intervals on which  $g$  is non-zero this  $g$  is  $0$  at isolated places that has to be dealt with separately all right as a next example we take a very popular problem from geometry this problem is is there in a variety of different books it's so popular that i really don't recall where i saw this example for the first time so i apologize for not giving you a reference for this find a plane curve what is the problem find the plane curve  $y$  equal to  $fx$  with the property that all its normals pass through the same point geometrically your intuition your geometrical intuition should tell you that this curve must be a circle well let us back up this intuition

with precise mathematical reasoning using calculus let us assume that the point through which all normals pass is the origin and let us take a typical point  $x$   $y$  on the curve with

equation  $y = f(x)$

so what is the slope of the normal at  $x$   $y$  well there is really no need for me to draw a picture because i urge you as you listen to these lectures i

urge you to doodle and draw the pictures yourself it is

so simple that there's no need for a

picture you also have your sharp imagination you got this point  $x$   $y$  on

the curve at the point  $x$   $y$  what is the slope of the tangent  $f'(x)$

so what is the slope of the normal normal is perpendicular to the tangent

so the slope of the

tangent is  $f'(x)$  the slope of the normal is going to be  $-\frac{1}{f'(x)}$

that is what you see in the slide  $-\frac{1}{f'(x)}$  to the power  $-1$

so what is

the equation of the normal the equation of the normal is  $y - y = m(x - x)$

equals  $m(x - x)$

so little rearrangement

gives you  $y - y = f'(x)(x - x) + x - x$  equal to

zero this is the equation of the normal now what are we saying we are saying that this

normal passes through the origin

so the origin satisfies this equation that's when i put  $x = 0$  and  $y = 0$  it should be this equation should be valid

so let us put  $x = 0$  here

and  $y = 0$  here what do we get we get the equation  $x + y$  of  $x$   $y$  equal to  $0$  this is equation 1.

13

so we see that 1.

13 must hold at all the points

on the curve at all the points in the curve we this equation 1.

13 must hold

so of course the

next thing to do would be to look at equation 1.

13 and simply drop these annoying subscripts

zero

so let's drop the annoying subscripts zero

so since this holds for all the points

$x$   $y$  on the curve and

so we write one point one three without the subscript

zero and using the familiar notation  $dy$  by  $dx$  in place of  $f'(x)$  and

so what is 1.

13  $x$

plus  $y \frac{dy}{dx}$  equal to  $0$  this  $y \frac{dy}{dx}$  equals  $-\frac{1}{x}$  it's a variable separable

differential equation it is a variable separable differential equation and you can immediately integrate it and you can get  $y^2 - x^2 = 2c$  the curve is the circle

so our intuition has been backed up by precise mathematical reasoning using calculus

so the curve is the curve with the property that all normals pass through a point is a circle

so now the next item in our agenda is to study a very interesting example from rainwell's book elementary differential equations the exact reference will be given later at the end of the example

so what are the example we take a variable separable equation  $\sqrt{1-x^2} dy + \sqrt{1-y^2} dx = 0$  of  $\theta = \arcsin y$  proceeding as usual we

get  $\int \frac{dy}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-y^2}} = 0$  the  $y$  integral as everyone knows is sine inverse of  $y$  and the  $x$  integral is sine inverse of  $x$  integral of  $0$  is of course a constant now

to determine the constant of integration we put in the initial data the initial conditions for the initial condition when  $x = 0$  what is  $y = \frac{\sqrt{3}}{2}$

so put  $x = 0$  here and  $y = \frac{\sqrt{3}}{2}$

so sine inverse of  $\frac{\sqrt{3}}{2}$  is  $\frac{\pi}{3}$  so value of the constant is  $\frac{\pi}{3}$

so that's about it i mean it's a very easy problem as far as solving this differential equation with this initial condition is concerned but we should not stop here we should

in fact take our investigation a little further and some very interesting things come up sorry while this may sound pretty easy let us see the solution in some detail

so what do we get value of the constant is  $\frac{\pi}{3}$  remember so we get sine inverse of  $y$  equals  $\frac{\pi}{3} - \text{sine inverse of } x$

so let's take the sine of both sides we get  $y = \sin(\frac{\pi}{3} - \text{sine inverse of } x)$  recalling the addition formula for sine

what is the addition formula for sine please it is  $\sin(a+b) = \sin a \cos b + \cos a \sin b$

and  $\sin(a-b) = \sin a \cos b - \cos a \sin b$  so this is going to

be  $\sin \pi/3 \cos \sin^{-1} x$  minus  $\cos \pi/3 \sin \sin^{-1} x$

of  $x$  all right

so that gives you  $y$  equal to  $\sqrt{3}/2 \cos \sin^{-1} x$  minus  $1/2 x$  then we should probably bring this  $1/2 x$  on the left hand side and square when

we square what do we get  $y + x/2$  the whole square is  $3/4 \cos^2 \sin^{-1} x$

inverse of  $x$  and which is  $1 - \sin^2 \sin^{-1} x$  what is  $1 - \sin^2 \sin^{-1} x$

inverse of  $x$  it is simply  $1 - x^2$  it is simply  $1 - x^2$  squared well the next thing

to do would be to expand this and collect terms well we get the more elegant formulation  $x^2$

squared plus  $y^2$  plus  $xy$  equal to  $3/4$ .

notice how different is this avatar of

the solution from our previous avatar our previous avatar was  $\sin^{-1} y + \sin^{-1} x = \pi/3$  and this now we got a very different

looking equation the solution  $y$  is implicitly given in terms of  $x$  let's proceed

a little further let us not give up at this stage let us try to understand what this curve

is it's a curve in the  $x, y$  plane if the  $xy$  term were not there had it just

been  $x^2 + y^2 = 3/4$  we'll be very happy and we'll be drawing a

circle of radius  $\sqrt{3}/2$  but unfortunately this  $xy$  term is going to complicate things

slightly but we would like to understand what kind of curve is this equation represent

$x^2 + y^2 + xy = 3/4$ .

to examine the nature of the curve  $x^2 + y^2 + xy = 3/4$

plus  $y^2$  plus  $xy$  equal to  $3/4$  let us put little  $y$  equal to  $1/\sqrt{2}$  upon

capital  $x$  plus capital  $y$  little  $x$  equal to  $1/\sqrt{2}$  into capital  $x$  minus capital  $y$  for those

of you who have learned the coordinate geometry properly will recognize that we are rotating

the coordinate system by angle  $\pi/4$  we are rotating the coordinate system through an

angle of  $\pi/4$  and trying to understand what happens to the equation  $x^2 + y^2 + xy = 3/4$ .

well the new equation the equation or the curve in

the new coordinate system is  $3x^2 + y^2 = 3/2$ .

low and behold this is

an ellipse  $3x^2 + y^2 = 3/2$  is a standard ellipse now let us look at

this ellipse a little carefully the ellipse in the new coordinates is  $3x^2 + y^2 = 3/2$ .

equal to  $3/2$ .

let us try to figure out what is the semi-major axis and semi-minor axis of this

ellipse let us divide the equation by  $3/2$  so as to make the right hand side 1

so if we do that

what happens let's divide by  $3/2$  what do we get  $x^2/3 + y^2/2 = 1$ .

so what is the semi-major axis the semi-major axis is square root of  $3/2$  the semi-minor axis is square root of  $2/3$  and the semi-minor axis is  $1/\sqrt{2}$ .

so the

standard ellipse the major and minor axis are along the coordinate axis but remember what did we do we rotated the coordinate system through angle  $\pi/4$  we rotated the coordinate system through angle  $\pi/4$ .

so what are the major and minor axis of the original ellipse it is the line sloping at an angle of  $\pi/4$  and the other line it's sloping at an angle of  $-\pi/4$ .

so and

so  $x^2/3 + y^2/4 + xy/2 = 1$  is a standard ellipse rotated by angle  $\pi/4$ .

so this beautiful example is from Earl D. Rainville's elementary differential equations the fifth edition this book has undergone many editions it has come to the tenth edition but i am referring to the fifth edition and the page numbers refer to the fifth edition

so this problem appears on page 4243 of this beautiful book and it's its remarkable collection of problems that i urge you to look at the birth of the elliptic functions now you might think that this particular problem was pretty simple it's very simple problem

$\int \frac{dy}{\sqrt{1-y^2}} + \int \frac{dx}{\sqrt{1-x^2}}$

root of  $1-x^2$  equal to 0 you might wonder that why do we do such trivial

exercise perhaps some of you might find this boring but i want to convince you that this

leads to something exceedingly non-trivial and very exciting part of mathematics the idea is to

replace this  $y^2$  by  $y$  to the power fourth and this  $x^2$  by  $x$  to the power four now if

you have been working out lots of integrals in your integral calculus classes you will know that

$\int \frac{dy}{\sqrt{1-y^2}}$  cannot be integrated you cannot

compute the indefinite integral of  $\frac{dy}{\sqrt{1-y^2}}$  to the power 4

well

so what happens is that whatever we had done we have got this equation sine inverse of  $x$  plus sine inverse of  $y$  equal to  $c$  in this elegant form  $x^2 + y^2 + x y = 3$

fourths so somehow you should probably get the idea that the differential equation is somehow

related to an ellipse second degree curve

so euler went further and he looked at this situation integral zero to  $u$   $u^4$  with with us fourth power in the denominator

so whatever we have done what we are just done can be slightly written in different form it is

the same idea but slightly in a slightly different avatar integral 0 to  $u$   $dt$  by root of  $1 - t^2$

square plus integral 0 to  $v$   $dt$  by square root of  $1 - t^2$  squared is again integral  $dt$  by square

root of  $1 - t^2$  square but the different limits 0 to some complicated expression involving

$u$  and  $v$  what is this complicated expression it is  $\phi(uv) = u \sqrt{1 - v^2}$

squared plus  $v \sqrt{1 - u^2}$  squared what euler did was he replaced the  $t$  squared by  $t$  to

the power 4 and obtained a similar expression with a more complicated fee and this was a very

remarkable achievement because three decades later gauss studied the inverse function remember

integral 0 to  $u$   $dt$  by root of  $1 - t^2$  square is sine inverse of  $u$  and its inverse

is the sine function similarly the function integral 0 to  $u$   $dt$  by square root of  $1 - t^4$

minus  $t$  to the power 4 also has an inverse called the elliptic sine function and those

were studied by gauss three decades later approximately in the year 1796 and gauss obtained

an addition formula for the elliptic sine function exact an analog of the addition formula for the

trigonometric sine function

so you see that what appeared to be a very innocuous looking differential equation has actually taken you through the gateway into a very magnificent part

of mathematics the theory of elliptic functions a very beautiful description of these things

can be found in a.

i markus shavish's book the remarkable sign functions for which

i give you the reference in the slides

so the idea was to take this  $x$  was to take this

simple variable separable differential equation and take it as an excuse to give you a

glimpse of a beautiful part of mathematics you