good morning welcome to this series of lectures on differential equations i am professor gopal krishna sunimasan from the mathematics department of iit bombay
 so we shall begin with these series of lectures with a short historical sketch we shall delineate some of the important things that are happened the beginning of this era of differential equations and calculus then i'll give you the names of a few important mathematicians were contributed to the study of differential equations
 so let's begin equipped with his calculus isaac newton with his laws of dynamics was able to explain the motion of the planets the precision of the equinoxes and the formation of tides astronomy which has hitherto in an empirical science has turned into a dynamical science this is a monumental achievement of newton which is why is considered to be very great not just for the invention of calculus but for the transformation of thoughts that the transformation of astronomy as a as a dynamical science in essence what newton did was he basically dealt with a system of differential equations for the two body problem and he was able to derive the kepler's laws of planet free motion
 so the origins of the theory of differential equations can be traced back at least to newton now let me get uh get you to uh the next slide indicating some some of the names of the masters the early masters were contributed to this subject first you see the name of isaac barrow who was a teacher of isaac newton some of the ideas of calculus already go back to isaac barrow then of course comes newton in 1687 he integrated a linear differential equation in 1693 you see the name of leibniz and he discovered that y equal to t x substitution for homogeneous equations then comes a very famous name of bernoulli we will encounter the bernoulli equation later maybe in the fifth or the sixth lecture and bernoulli has contributed phenomenally to the theory of differential equations then you see the name of riccati which is famous equation y prime equal to ax y squared plus bxy plus cx this differential equation is notorious because it cannot be solved innocent though it looks it cannot be solved except in special cases leonard euler the great genius he showed how the riccati equation can be reduced to a linear equation if a certain solution is known the list is very long and we must truncate this list because the subject of differential equations is at least 350 years old and we need to get to the nitty-gritty of this present course and we cannot dwell any longer on this historical development i'd like to just give you one reference for the historical developments the opening chapter of this book by rabbi on differential equations an introduction to basic concepts results and applications the second edition it has a beautiful historical introduction laws of physics when expressed in mathematical terms give rise to differential equations for example the law of mass action in chemistry gives you the differential equations of chemical kinetics and enzyme kinetics models arising in biology ecology demography they all give rise to differential equations we shall see some of these examples coming from ecological models in the later part of today's lecture one sees some striking similarities between the differential equations that are that are arise in chemical kinetics and mathematical ecology there are strong similarities between the two kinds of systems mathematical biology has today grown into a vast area and active research is going on in mathematical biology for example ec siemens succeeded in modeling of the heartbeat as a system of differential equations hodgkin and huxley for their work of neural impulses received the nobel prize then there are problems which arise in geometry
 so here is a situation where we see application of one part of mathematics to another part of mathematics
 so differential equations are all over the place in the physical sciences in

engineering in biological sciences in chemistry ecology demography spread of diseases growth of tumors and a whole lot of other things and yes problems in geometry give rise to differential equations

so there is plenty of reasons why one must study differential equations there are persuasive reasons why one must study different differential equations and what you will see in this course is just the beginning a modest beginning of a very vast area of mathematics okay

so let's begin by looking at a very simple physical situation the simple pendulum

so what is a simple pendulum consists of it consists of a bob of mass m suspended from a point as you see in this picture a bob of mass m suspended by a weightless rod of length l which is set into oscillations and this angle this is the mean position or the bob and this bob is displaced by an angle y and it is set in oscillations you see that the mass of the bob is m and mg is the weight acting vertically downwards and mg sine y is the component is the component of this direction the tangential direction now we would like to show how the newton's second law of motion gives rise to the system of differential equations governing the motion of this bob of the pendulum

so let's see the angular displacement at time t is y of t okay and now we observe that the angular acceleration

so if the angle of displacement is y then what is the angular velocity it is d y by dt what is the angular acceleration it is d2y by dt squared and you've got a force which is acting vertically downwards and that force is going to give rise to a torque

so what is the magnitude of this torque the magnitude of this torque is mgl sine y you have to multiply this by l to get the torque and now what is the moment of inertia of this bob the moment of inertia of this bob is m l squared

so let's go back to the slides and you see the moment of inertia is ml squared you multiply the angular acceleration d2y by dt squared by the moment of inertia mn squared and that will be balanced by the external torque an external torque is minus mgl sine y the m cancels out and you get equation 1.
1 d2y by dt squared plus g over l sine y equal to 0.

so equation 1.
1 is the differential equation governing the motion of a simple pendulum

so this simple pendulum

so the motion of the bomb or the simple pendulum is governed by this differential equation d 2 y by d t squared plus g over l sine y equal to 0 that is the differential equation for the motion of a simple pendulum simple pendulum is a ma of mass m of length l and set into motion okay

so let's get back to this differential equation 1.
1

so you see that there are two time derivatives it is d2y by dt squared

so this is a second order differential equation all right and um

so let's go go to the next one

so let us look at one more example from physics shm stands for simple harmonic motion

so what is a simple harmonic motion a particle is said to exhibit a simple harmonic motion if it moves along a straight line first of all it moves in a straight line and the force acting on the particle is proportional to the displacement from the origin and and the force acts in a direction opposite to the displacement

so the displacement of the particle is y of t then the acceleration is d2y by dt squared and multiply this acceleration by m you get the force and this force

is proportional to y and

 so this force has magnitude k y and it is going to pick up a negative sign because the direction is opposite

 so the balance law gives you m d2 y by dt squared plus ky equal to 0 divide by m and call k by m as omega squared and we get the differential equation 1. 2 d2y by dt squared plus omega squared y equal to 0.

 so equation 1.
2 is the second order differential equation why is it a second order differential equation because you see that the second derivative appears d2y by dt squared all right

 so so now here we see a second example from physics where we where we got the differential equation by balance law by looking at a balanced law

 so i again repeat that laws of physics when expressed in mathematical terms give rise to differential equations and we already seen two such examples two examples both of them are second order differential equations well let's move on a little more a little further and again look at examples from physics but before that let us look at this simple harmonic motion in a little bit of detail

 so the equation of simple harmonic motion is again displayed in the slide d2y by dt squared plus omega squared y equal to 0 anybody can substitute cosine of omega t into the equation 1.
2 and directly verify that cosine of omega t satisfies equation 1.
2

 so we call cosine of omega t as a solution of equation 1.
2 similarly one could try y equal to sine omega t substitute it into the differential equation 1.
2 and one can verify that sine omega t is also a solution of equation 1.
2

 so we got two solutions cosine omega t and sine omega t of 1.
2 now in physics you are familiar with the idea of superposition

 so you take superposition of two waves right

 so what is the mathematical meaning of superposition what does it mean to say that we take a superposition of a cosine and a sine it means you look at that a third type of solution 1.
3 namely a cos omega t plus b sine omega t

 so let us take equation 1.
3 and substitute into equation 1.
2 and you will be able to verify quickly that 1.
3 also satisfies the differential equation 1.
2

 so what have we done we have now enlisted many solutions of 1.
2 namely cosine omega t sine omega t and generally a cosine omega t plus b sine omega t that is 1.
3 notice that in 1.
3 if you take a equal to 1 and b equal to 0 we get cosine of omega t if we take a equal to 0 and b equal to 1 we get sine omega t and i could give a various values 1 2 3 minus half 1 3 whatever and you could give b various values 1 upon root 2 1 minus 1 0 etc

 so for every choice of constants a and b equation 1.
3 displays a solution of the harmonic oscillator equation 1.
2

 so we have enlisted many solutions of 1.
2 in fact we enlisted an infinite family of solutions of 1.
2 however let us ask ourselves this question have we enlisted all the solutions

how do i know that this 1.

3 exhausts all the solution maybe somebody can be exceedingly clever and produce a solution of 1.

2 which is not of the form a cosine omega t plus b sine omega t this question must be answered that is how do we know that if zt is any solution of 1.

2 then zt is of the form a cosine omega t plus b sine omega t for certain constants a and b we see that we are naturally led to the problem of describing the class of all solutions it is not difficult to show that 1.

3 exhausts all the solutions of 1.

2 every solution of 1.

2 is of the form a cosine omega t plus b sine omega t this is not difficult to prove but we shall not do that at the moment we shall come back to this much later if time permits we shall instead move on to some more examples let us look at some examples of electrical circuits what is the analog of 1.

2 1.

2 is a mechanical a system of a simple harmonic motion of a particle moving along a straight line by a force which is linear and in opposite direction analog of 1.

2 also arises in the theory of electrical circuits namely the lc circuits the l stands for inductance and c stands for capacitance

 so for a discussion of these lc circuits i shall refer you to the famous book of robert resnick and david halliday which i'm sure you're all reading currently for your preparations and the second volume for the third edition please pay attention to the edition because this book has undergone several editions

 so if you pick up the wrong edition we will not be able to we are not on the same page

 so i'm talking about page 845 equation 38.

5 in the third edition of wrestling and holiday famous book of physics volume 2.

 there you will see a very detailed description of this electricals lc circuits in fact he talks about the lcr circuits on page 848 and what are the equation governing the lcr circuit it is d2 q by dt squared plus r over l dq by dt plus q over lc equals zero here the r is a resistance l is the inductance and c is the capacitance in particular if the resistance is zero in particular the resistance is zero what happens to the differential equation you see the r term the middle term is not there the middle term is just not there if r is zero

 so what do you have you got d two cubed by dt squared plus constant times q is zero what is the constant one upon lc one upon lc is a positive constant

 so you can call it omega squared

 so you see d2 cube by dt squared plus omega squared q equal to 0 but isn't that the same as 1.

2 1.

2 reads d2y by dt squared plus omega squared y equal to zero

 so you see that the lc circuits that you encounter in electrical circuit theory is electrical analog of the mechanical system of a simple harmonic motion

 so the differential equation is very similar on the other hand if you throw in a resistance term then you have a middle term for the middle term the r by l dq by dt is the middle term

 so this equation would be a oscillator with a damping term thrown in

 so i'll refer you to equation 15.

37 of wrestling and holiday now let us leave the realm of physics and slowly move to the realm of biology specifically mathematical ecology way back in 1798 malthus proposed a model for the population growths of an ecology in which there's only one species of organism for example you can think of bacteria the growth of bacterial cultures now there is only one species in this ecology this

model was also proposed independently by leonard euler somewhat earlier what the model say it says that if y of t is a population of the species at time t when the rate of change of population d y by dt is proportional to the population present at the time in other words dy by dt is k times y for some constant k of proportionality this is equation 1.

4 is the equation of a single species ecology or the malthusian model note that 1.

4 immediately gives you that y of t must be a e to the power kt the equation 1.

4 is satisfied by a e to the power kt a times exponential of kt

 so this shows that the population yt must grow exponentially fast

 so according to the malthusian model for a single species ecology the population yt explodes it grows very very fast it is e to the power kt let us ask ourselves a couple of questions is this practical can population really grow exponentially fast over an unlimited amount of time would nature allow such an exponential growth of population would there not be some inhibiting factors caused by say limitations of natural resources which will prevent this exponential growth we know that this cannot continue the population cannot keep on growing exponentially fast at some point of time there must be some mechanism by which this is stopped such a mechanism was proposed by verhust in 1836 what varhool says is that look at the differential equation 1.

4 there is just the d y by dt equal to ky what what says is that the right hand side must be modified you see when there is paucity of natural resources when there is positive food then that causes social friction that causes social friction and that social friction is going to get have a negative effect on the growth of population therefore the ky term must be changed to a k y minus k y squared by r term look at equation 1.

5 the differential equation has changed the rate of change of population is not d y by dt equal to ky it is d y by d t equal to k y minus some term what is the term k y squared by r where r is another constant this r the second constant r is called the environment's carrying capacity this constant r will depend upon the limitations of the environment in which the ecology is evolving there is a very beautiful book on mathematical biology by j.

d murray i have given you the reference for that it's one of the most comprehensive treaties written on mathematical biology and you will find in this book lots of historical details the various models the various models that various scientists have proposed the merits and demerits of these various models the kinds of differential equations that you get by making various assumptions and

 so on now let us go to another another system let us go to an ecological system but this time it this ecology contain contains two types of species a predator and a prey for example you can think of the predator as sharks and the prey as sardines for instance or any predator and any prey you can think of cats and mouse if you like

 so let us consider two species ecology consisting of a predator with population x d and a prey with population y t

 so for the predators the food the source of food is availability of the prey and the prey they are herbivorous animals they are they for example if you are looking at aquatic system then you assume that the prey lives on algae for example natural vegetarian food

 so suppose for example there is no prey suppose y is not there then the predators don't have food to eat

 so their population will die down exponentially fast look at equation 1.

6 dx by dt equal to minus ax that means that the exp that means that the function x t is going to be like e to the power minus a t it is going to decay

as time goes off to infinity in other words if the if there are if there are no prey if yt is not there then the rate of population of x t will keep declining very fast on the other hand if there are no predators there is an xt is not there then there is nothing to stop the growth of population of the prey the prey are herbivorous animals and their population keeps on increasing exponentially fast for example you can think of an environment with rabbits and foxes for example the the prey could be rabbits and the predators could be foxes the rabbits are herbivorous animals and the foxes are carnivorous now let us look at let us put these two species of animals together

so the rate of decrease

so 1.
6 is no longer true because the predators now have food to eat

so the decline in population that is the term minus ax is going to be checked by at the addition of a plus b x y term you see the last line in the slide ah the equation the right hand side of equation 1.
6 is modified by replacing minus ax by minus ax plus bxy mind you all the constants a b c and k are all positive okay now what will happen to the and the rate of increase indicated in the right hand side of 1.
7 now you put the predators and pray together now that the predators are there they're going to eat the prey

so the population of prey cannot keep on increasing like 1.
7 the rate of population increase d y by dt equal to ky the ky term is going to be modified how is it going to be modified you're going to be looking at k y minus c x y you'll be going to be looking at k y minus c x y and

so the modified system of equations will be dx by dt equals minus ax plus bxy dy by dt equal to ky minus cxy you get a pair of differential equations 1.
8 constants a b c and k are on positive now at this juncture you might ask why did i put a plus b x y and a minus c x y why this quadratic term x y y naught x squared y y naught x y squared well these are models and they are not accurate models first of all secondly let us do a thought experiment suppose you have an environment with say it's a cats and mice all over the place now suppose the population of cats doubles and the population of mice also doubles then the probability of him of a cat encountering a mice is going to go up by four times then that explains why the x y term and this interaction between a cat and a mouse is going to be detrimental to the mouse whereas it's going to be favorable to the cats

so that's when the first equation you see this term with the plus sign and the second equation you see this term with a minus sign

so that sort of empirically explains this model 1.
8 this model is a very famous model it is called the volterra lotka model it's a volterra loadcast system of equations it's a system of simultaneous differential equations and it's a coupled system of differential equations remember look at the first equation in 1.
8 it is dx by dt equals minus ax plus bxy the y appears in the first equation also and the second equation is d y by dt equal to ky minus cxy the x appears in the second equation

so the system of equations are a couple the system of equation namely both unknowns x and y appear in both equations and this is a simultaneous system of differential equations and this example i given just to illustrate how differential equations arise in the study of ecological systems we are not going to study we are not going to analyze this equation 1.
8 in detail because that's not quite in the scope of this course but the this example has just been put here to show that differential equations are all over the place again they appear in physics as in electrical circuits mechanical

systems they appear in the growth of bacteria and population models demography spread of diseases chemical kinetics and a lot more and a lot more all right
 so i would like to close this brief discussion of biological systems how differential equations arise in biology i spoke for quite some time on this
 so it only makes sense to give you some references which you can read at some point of time there are hundreds and hundreds of books written on mathematical biology and i have selected three of them the the last one is a very nice book by jd murray which i already mentioned and the first book ds jones m j planck and bd sleeman says differential equations in mathematical biology this book gives you large number of systems of differential equations arising in various biological problems such as the growth of tumor for instance spread of diseases and many other biological systems have been discussed biological differential equations arise in physiology blood flow in the aorta that gives rise to interesting systems of differential equations and a whole book a very fat volume has been written on mathematical physiology by keener and schneid of course we are not going to say much more about these things
 so this is a introductory uh lecture on how differential equations arise the genesis of differential equations the origins of differential equations
 so now let us come to specific differential equations that that are relevant for this course
 so now on we are going to study only differential equations in the first order how do how does the differential equation of first order look like d y by d x equals f of x comma y equation 1.
9 that you see on the slide this is a first order differential equation this is the first order differential equation because only one derivative appears d y by d x in contrast the shm the simple harmonic motion the equations governing the simple harmonic motions the equation governing the motion of a pendulum the lcr circuits inductance capacitance resistance all these systems involve differential equations with two derivatives
 so there are second order differential equations the study of before we take up the study of second order differential equations it is natural to begin with first order differential equation the easier ones and in this course that is all that we will be looking at for second order differential equations you need to look at look beyond the present series of lectures
 so f of x y is a function of two variables
 so we shall primarily be looking at three types of differential equations 1.
9 namely those differential equations which are called the variable separable equations and the second type of differential equations is known as the homogeneous differential equations and the third is the linear equations and their close cousins namely the bernoulli equation remember that right to the beginning i put it put down the name of jean bernoulli who was a very important figure in this historical development of differential equations it is the same bernoulli
 so his name appears now
 so linear equations and bernoulli equations
 so we shall look at these three types of equations in this course ok
 so let us take up the first one the variable separable equation all right
 so the variable separable equation
 so this is a very special type of differential equation where the right hand side f of x y right hand side f of x y could be a variety of different functions right f of x y could be x squared plus y squared or f of x y could be x into y f x y could be sine x into cos y for instance
 so the these last two cases are very special what is special about it the f of x y is a function of x times a function of y but in the first example that i

gave you f of x y equal to x squared plus y squared this is not a product of a function of x and a function of y correct

so but what are the variable separable equation the equation 1.
9 is a variable separable equation if f of x y is of the form as of h x into g y that is sorry sorry g x into h y

so the the k the case when the function f of x y appearing in 1.
9 is a product of the form g x into h y then we say that the differential equation is variable separable right okay

so in 1.
9 f of x y is gx into h y right it is going to be g x into h y let's write it let me write that down

so you are looking at d y by d x equal to f of x y and i'm assuming that this f of x y is of the form g x into h y

so what am i supposed to do next i will divide by h of y and i will write it as 1 upon h of y into d y by dx equal to g of x of course i am going to assume that h of y is not 0 i'm going to make the assumption what then what's next that's a function of x and y the solution is also a function of x and h is a function of y and y in turn is a function of x i'm going to integrate both sides with respect to x

so you see equation 1.
10 had divided by h i'm going to assume h y is not zero in fact we're going to assume that neither g x is 0 nor h y is 0.

what happens when one of them is 0 that we shall take up later and such cases will always arise in any mathematical analysis when you divide you will have to look at those special cases where the quantity by which you divide is zero those cases are always dealt with separately we shall do that later

so let us not interrupt the flow of argument

so we divide by h of y and you integrate both sides of equation 1.
10 with respect to x what do you get you get 1 upon h y d y by d x into dx equals integral gx dx now look at the left hand side the integral on the left hand side is very special it is very tempting to apply the uh substitution theorem it is very tempting to apply the substitution theorem

so you

so that y of x remember h of y and y is a function of x y is a function of x you want to put y of x equal to u if y of x equal to u then d y by dx dx will be d right

so our integral on the r will transform to d y by h y equals integral g x dx

so there's a substitution theorem but before we apply the substitution theorem you should be careful that the derivative should not be zero the change of what is the substitution theorem you are making a change of variables right for example recall in your integral calculus you want to integrate square root of 1 minus x square dx you will say put x equal to sine theta well you can put x equal to sine theta as long as you are working in the interval minus pi by 2 pi by 2 the change of variables x equal to sine theta the derivative must be non-zero in other words d y by dx must be non-zero go back to the differential equation 1.
10 we are already assuming h y is not zero now we are going to assume g x is not zero

so your d y by d x is going to be non-zero the substitution theorem is valid and

so we get equation 1.
11 okay now what remains we need to integrate gx dx on the right hand side and we need to integrate d y by h y on the left hand side hopefully we can perform these integrations and hopefully we can get a closed answer but you know from

your from your experience that these are only hopes and they cannot always be realized you know often many examples of functions whose indefinite integral g x d x cannot be computed or its computation can be pretty tricky sometimes they are easy we are lucky if they are easy to compute these integrals are easy to compute often we are not
 so lucky either it requires some very clever manipulations and sometimes it's just not possible to compute the indefinite integral but that's life we have to accept it let us look at a simple example and usually what happens is a differential equation comes equipped with some initial conditions well i'll come to the solved example after this comment we think of the variable the independent variable as time right and the dependent variable is say the population at time time t and we have seen or it could be the current flowing through an electrical circuit or it could be the displacement from the mean position of simple pendulum
 so what we need to do is that we need to think of the independent variable as a time variable and at some point of time say time t equal to t naught you need to be given the data such as the population at time t equal to t naught the angular displacement at time t equal to t naught or the concentrations of chemical reactants in a chemical if you are studying chemical reaction then the concentration of the various substances reacting substances they need to be specified at a certain time t naught or the current if the if the studies of an electrical circuit then the current at time t equal t naught may be specified
 so you need data at time t equal to t naught that is the the solution that you are trying to find should be prescribed at at least one particular point of time say time t equal t naught in other words we are given the value of y at time t equal to t naught that you are given the value of the solution y t the solution y t is prescribed at time t equal to t naught and what your what you do is you look for the solution in an interval t naught minus a to t naught plus a
 so normally differential equations come equipped with certain side conditions such as initial conditions in other words you're looking at you're not only looking at the differential equation you're not only looking at the differential equation $\frac{dy}{dx}$ equal to f of x y this is supplemented with some extra condition like y of x naught what the value of the solution y at x equal to x naught is given to you say y naught there's one piece of warning that you see even in the differential equation is defined everywhere the interval i on which the solution makes sense may be limited let us see this in a very simple and special case let us look at the differential equation $\frac{dy}{dt}$ equal to y squared is a very innocent looking differential equation $\frac{dy}{dt}$ equals y squared and let us say that the value of the solution at time t equal to 0 is c where c is a constant and assume that c is a positive constant just to fix ideas then proceeding as above what do we what do you mean proceeding as above we divide by y squared and integrate both sides with respect to t right we divide by y squared and integrate both sides with respect to t
 so let's do that
 so we're looking at $\frac{dy}{dt}$ equal to y squared y of 0 equal to c
 so we divide by y squared 1 upon y squared $\frac{dy}{dt}$ equal to 1
 so integrate with respect to time t both sides get integral 1 upon y squared d y by dt dt equals integral dt right that's exactly what we did now this this one the left hand side will simplify to integral d y by y square thanks to the change of variables substitution theorem substitution theorem i'll give you this that's exactly what you see in the slide you see d y by y squared equals dt integrate and remember the constant of integration these are indefinite integrals
 so there will be that constant of integration floating around

so let's do that let's put in the constant of integration and see what happens
so you're looking at integral d y by y squared equal to integral dt
so what happens minus 1 upon y equals t plus b where b is the constant of integration correct
so that reads y of t equal to minus 1 upon t plus b right now put in the initial conditions remember that we are given initial conditions put t equal to 0 what do we know when t equal to 0 we know that y of at t equal to 0 is c
so what do we get c equal to minus 1 upon b or b equal to minus 1 upon c the value of the arbitrary constant has now been computed we have calculated the value of the arbitrary constant integration constant all right now let us feedback into this put this value back here what do we get we get y y of t equal to minus 1 upon t minus 1 upon c or will be c by 1 minus c t
so that's the solution of the differential equation that is the solution of the differential equation that's a function notice that what happens if t approaches 1 upon c what happens when t approaches 1 upon c what happens to this object here this thing blows up to infinity that thing blows up to infinity
so here you see this in the slide i had commented this in red as t tends to 1 by c from the left that is t starts at the origin and it progresses and time evolves and when time t approaches 1 by c what happens to the solution the solution simply goes off to infinity the solution escapes to infinity in finite amount of time
so the solution the physical system makes sense the evolution of the physical system makes sense only up to time t equal to 1 by c beyond t equal to 1 by c the physical system has already exploded there's been a catastrophe okay
so the time the interval on which the solution exists is not the whole real line it is all the way from minus infinity up to 1 by c not beyond this is what the caption says the solution escapes to infinity in finite time solution escapes to infinity in finite time
so so let me just go back to the previous slide and now let us look at the warning even if the differential equation is defined everywhere look at the differential equation the differential equation is d y by dt equal to y squared there is nothing wrong with the differential equation y squared is defined everywhere the differential equation is defined everywhere nevertheless the solution escapes to infinity in finite time
so even the differential equation is defined everywhere the interval i on which the solution makes sense may be limited and that is exactly what happens in this very special case
so we'll stop this first lecture here and we'll continue this in the second lecture thank you very much
you