

hello viewers welcome to lecture 6 on integral calculus today we will do some more problems on differential equations

So let us start with question number one let f be a differentiable function on \mathbb{R} such that $f'(x) = 1$ if the y intercept of the tangent at any point (x, y) on the curve $y = f(x)$ is equal to the cube of the abscissa of (x, y) then find the value of f at minus three

So here we are given some condition we have to first form a differential equation and then solve it

So we know that slope of tangent at (x, y) is given by $\frac{dy}{dx}$

So therefore equation of the tangent at (x, y) is given by $y - y = \frac{dy}{dx}(x - x)$

So here because the point is given as (x, y) we have used x and y to write the equation of the line

So putting $x = 0$ we get $y = y - x \frac{dy}{dx}$

So this gives the y intercept and abscissa of this point (x, y) is x

So what is given is the y intercept $y - x \frac{dy}{dx}$ this is equal to x^3

So this can be rewritten as $\frac{dy}{dx} - 1 = x^2 y$

So we divide the equation by y to get this now this is a linear first order ode

So we know to solve this we need to find integrating factor equal to exponential of integral of $\frac{1}{y} dx$ which is $e^{\int \frac{1}{y} dx} = e^{\ln y} = y$

So the solution is $y \cdot y = \int 1 \cdot y dx$

So that is $y^2 = \int y dx$

So $y^2 = \frac{1}{2} x^2 + c$ this implies $y = \sqrt{\frac{1}{2} x^2 + c}$ now we are given that $f'(0) = 1$

So when $x = 0$ $y = 1$ this implies $1 = \sqrt{c}$ which implies $c = 1$

So therefore y is given by $y = \sqrt{\frac{1}{2} x^2 + 1}$ now we can calculate f at minus 3 equal to $\int_{-3}^0 \sqrt{\frac{1}{2} x^2 + 1} dx$ which is equal to $\frac{1}{2} \int_{-3}^0 \sqrt{x^2 + 2} dx$

So that gives 9.

So this is the answer to this problem ok now let us move to question number two question two says that let γ denote a curve $y = f(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it now let the tangent to γ at a point (x, y) intersect the y axis at point $(0, p)$ if $|p - y|$ has length one for each point (x, y) on γ then which of the following options is or are correct

So if you see the options then option a and option c are negative of each other and we are given that the curve should lie in the first quadrant

So y should be bigger than zero

So only one of the two options a and c can be correct of course it may happen that both are wrong

So let (x, y) be the point whose coordinates are (x, y) then in the first problem we have derived the y intercept at a general point on a curve

So we know that $(0, p)$ is the point $(0, y - x \frac{dy}{dx})$

So by the previous problem now what is given is given the length $|p - y|$ is equal to one

So we get $1 = |y - x \frac{dy}{dx} - y|$ and the distance of (x, y) squared will be $x^2 + y^2$

So this gives $\frac{dy}{dx} = \frac{1 - y}{x}$ or $\frac{dy}{dx} = \frac{1 + y}{x}$ now since we have a square of $\frac{dy}{dx}$ this has to be bigger than or equal to 0 this implies that x^2 must be less than equal to $1 - y^2$ which means that x has to be between minus one and one

So therefore we will get that $\frac{dy}{dx}$ at any point on the curve is equal to square root of $1 - y^2$ divided by x with plus or minus sign

So this is because γ lies in the first quadrant we have $x > 0$

So x is positive x will lie in the open interval $(0, 1)$. now here we have to be careful and see which sign is valid note that it may happen that this $\frac{dy}{dx}$ is with positive sign for some x in the interval $(0, 1)$ and it may be negative for some x but we will show that that is not possible why is that because if $\frac{dy}{dx}$ is greater than zero for some x one in zero one and $\frac{dy}{dx}$ is negative for some x in zero one then by the continuity $\frac{dy}{dx}$ must be zero at some x in zero one however $\frac{dy}{dx} = \frac{1 - y^2}{x}$ we got is $\frac{1 - y^2}{x}$ by x^2 this is non zero for all x in the open interval zero one

So therefore either $\frac{dy}{dx}$ is equal to square root of $1 - x^2$ by x or $\frac{dy}{dx}$ equal to minus of square root of $1 - x^2$ by x for all x in the interval $(0, 1)$. So now if $\frac{dy}{dx}$ is greater than 0 in $(0, 1)$ then we know that the derivative greater than 0 implies that $y(x)$ is an increasing function in the interval $(0, 1)$ also it is given that $y(1) = 0$ because $(1, 0)$ lies on γ . So this implies $y(x)$ must be negative in the interval $(0, 1)$ but if $y(x)$ is negative in $(0, 1)$ this will imply that γ lies in the fourth quadrant this is not true. So therefore $\frac{dy}{dx}$ is with negative sign this is negative of $\frac{\sqrt{1 - x^2}}{x}$ for x belonging to $(0, 1)$ this immediately gives that $x \cdot y' = -\sqrt{1 - x^2}$ plus square root of $1 - x^2$ this must be equal to zero. So therefore option b is correct and option d is incorrect. So we got b is correct now we have to see whether a or c is correct or not. So now we have $\frac{dy}{dx} = -\frac{\sqrt{1 - x^2}}{x}$ is equal to minus square root of $1 - x^2$ by x . So by integrating we get y is equal to minus integral of square root of $1 - x^2$ by $x \, dx$ and this integral is easy to evaluate. So what we can do is we can put $x = \sin \theta$ then $\frac{dx}{d\theta} = \cos \theta$ and square root of $1 - x^2$ will be equal to $\cos \theta$. So this y is equal to minus integral $\cos \theta$ by $\sin \theta$ into $dx = \cos \theta \, d\theta$. So that is equal to minus of $\cos^2 \theta$ which is $\sin^2 \theta - 1$ by $\sin \theta \, d\theta$. So that is equal to integral of $\sin \theta \, d\theta$ minus integral $\cos \theta \, d\theta$ which is equal to $-\cos \theta + \log |\cos \theta| + \cot \theta + \text{an arbitrary constant } c$. So now if we put $\sin \theta = x$ $\cos \theta = \sqrt{1 - x^2}$ this will give this implies y is equal to minus square root of $1 - x^2$ plus $\log |1 + \sqrt{1 - x^2}|$ by x plus c we do not need to put the modulus sign because x is positive and since $y(1) = 0$. So this will imply $c = 0$. So therefore y is equal to minus square root of $1 - x^2$ plus $\log |1 + \sqrt{1 - x^2}|$ by x . So if we go back to the options then we see that option a is correct and option c is incorrect. So note that in this problem getting the differential equation was not difficult and also integration was not difficult the only tricky part was to notice that only one of the signs of $\frac{dy}{dx}$ is possible if you are not careful about that then you might think that all the options are correct. So let us move to question number three a solution curve of the differential equation $x^2 + xy + 4x + 2y + 4 \frac{dy}{dx} = y^2$ for x greater than 0 passes through the point $(1, 3)$ then this solution curve a intersects $y = x + 2$ at exactly one point b intersects $y = x + 2$ at exactly two points c intersects $y = x + 2$ square d does not intersect $y = x + 3$ square. So first we have to solve this differential equation. So we are given that $\frac{dy}{dx} = \frac{y^2 - x^2 - xy - 4x - 4}{4x}$ now if you notice the denominator can be factorized as we have $x^2 + 4x + 4 = (x + 2)^2$ $xy + 2y = y(x + 2)$ that is $y(x + 2)$. So this is y^2 divided by $(x + 2)^2$ minus y by $x + 2$ plus $\frac{4x + 4}{4x}$. So now we see that this $\frac{dy}{dx}$ equal to the sum function of $\frac{y}{x + 2}$ and y . So this can be written as just like we do for homogeneous equation. So this can be written as $y = u(x + 2)$ divided by $1 + u$ by $x + 2$. So now to solve this we can put $y = u(x + 2)$ this will imply $\frac{dy}{dx} = u + x + 2 \frac{du}{dx}$ and therefore we have $u + x + 2 \frac{du}{dx} = \frac{u^2}{1 + u}$. So this implies $x + 2 \frac{du}{dx} = \frac{u^2}{1 + u} - u$ which is equal to $-\frac{u}{1 + u}$. So now we can separate the variable u and x . So this can be written as $1 + u \frac{du}{u^2} = -\frac{1}{x + 2} dx$ then we integrate both sides this implies $1 + \frac{1}{u} = -\log |x + 2| + c$. So that is $u = -\log |x + 2| + c$. So this implies we can bring this log this side and we get $\log |u(x + 2)| + u = \text{constant}$ now we know that $u(x + 2)$ is y .

So this implies \log of $\text{mod } y$ plus u is y by x plus two equal to c also we are given that this curve passes through the point one comma three

So y at one is equal to 3 this implies $\log 3$ plus 3 by 1 plus 2 equal to c which implies c is equal to one plus $\log 3$

So the solution curve is given by $\log \text{mod } y$ plus y by x plus two equal to one plus $\log 3$

So this is the equation of the solution curve now let us look at the options the first option is asking whether y equal to x plus two intersects this curve and if

So then now at how many points

So putting y equal to x plus 2 we get $\log \text{mod } x$ plus 2 plus 1 equal to one plus $\log 3$ which implies \log of $\text{mod } x$ plus two equal to $\log 3$ and since x is greater than zero we can simply write \log of x plus two equal to $\log 3$ and the only x for which this happens is x plus two equal to three that is x equal to one

So therefore we have that the solution curve intersects y equal to x plus two at exactly one point

So a is correct and b is wrong now we have to see whether it intersects the curve y equal to x plus 2 square or not

So putting y equal to x plus 2 square in let us call this equation star we get $\log y$ So \log of x plus 2 square plus y by x plus 2 x plus 2 square by x plus 2 equal to one plus $\log 3$ which is same as $2 \log x$ plus 2 plus x plus 2 equal to one plus $\log 3$

So we have to see whether there exist any x for which this happens or not for any x greater than 0 this left hand side is greater than x plus two will be bigger than two and $2 \log x$ plus two will be bigger than two $\log 2$

So two plus two $\log 2$ is same as two plus $\log 4$ which clearly is bigger than one plus $\log 3$

So therefore lhs is not equal to rhs for every x greater than 0

So therefore the curve does not intersect y equal to x plus 2 square option c is wrong now we have to see whether it intersects the curve y equal to x plus 3 square now putting

y equal to x plus 3 square in star we get \log of x plus 3 square plus x plus 3 square divided by x plus 2 equal to one plus $\log 3$ that is $2 \log x$ plus 3 plus x plus 3 square by x plus two equal to one plus $\log 3$ now again since x is bigger than zero $2 \log x$ plus three plus x plus three square by x plus two this will be bigger than the first term is bigger than two $\log 3$ and x plus three square by x plus two is clearly bigger than one because x plus three is bigger than x plus two

So again therefore this is not possible does not hold for any x bigger than 0 therefore the solution curve does not intersect y equal to x plus 3 square

So that means that option d is correct ok now let us move to question number four let a solution y equal to y of x of the differential equation x times square root of x square minus 1 dy minus y times square root of y square minus 1 dx equal to 0 satisfy y at two is equal to two by root three now consider two statements one is y/x is equal to secant of secant inverse x minus π by 6 and second statement is y/x is given by 1 by y is equal to $2 \sqrt{3}$ by x minus square root of 1 minus 1 by x square then we have 4 options both 1 and 2. are true b one is true but two is false c is one is false but two is true and d option is both the statements 1 and 2 are false

So we are given some first order ordinary differential equation with initial condition y at two is equal to two by root three and then we have to see which one can be a solution to that

So first thing of course you can try and see by putting x equal to 2 whether you are getting y equal to two by root three or not if you try that then you will see that both these statements satisfy that initial condition y of 2 equal to two by root 3

So that does not help

So we will try to solve the differential equation and see what solutions we are getting So listen what we are given is dy by y times square root of y square minus 1 is equal to dx divided by x times square root of x square minus one

So if you note that we know that the derivative d by dx of secant inverse x this is equal to 1 by $\text{mod } x$ times square root of x square minus 1. now here we have dx by x times square root of x square minus 1

So if x is positive then of course the integral will be secant inverse x but if x is negative then we can just multiply the equation by negative one and we will still get

So we do not have to worry about that and by integrating we get secant inverse of y is equal to secant inverse of x plus constant c now we will find the value of c by using the

condition that y at 2 is equal to 2 by root 3

So y at 2 is equal to 2 by root 3 this implies secant inverse of 2 by root 3 is equal to secant inverse of 2 plus c which implies c is equal to secant inverse of 2 by root 3 will give π by 6 and secant inverse of 2 is π by 3

So π by 6 minus π by 3 which is equal to minus π by 6 .

So therefore we get secant inverse of y is equal to secant inverse x minus π by 6 which implies y is equal to secant of secant inverse x minus π by 6 .

So we see that the statement 1 is true now we have to see whether statement 2 is true or false

So in statement 2 we are given 1 by y

So if we see we know y is equal to secant of secant inverse of x minus π by six this is correct

So this will imply one by y is equal to cos of secant inverse x minus π by 6 and then by using the formula for cos of c minus d we get this is cos of secant inverse x times cos of π by 6 plus sine of secant inverse x times sin π by 6 which is equal to cos of secant inverse x will be 1 by x and cos π by 6 is root 3 by 2 plus sine of secant inverse x is square root of 1 minus 1 by x square times sine π by 6 will be equal to half

So you can put secant inverse x is equal to θ that will mean x is equal to secant of θ

So cos θ will be equal to 1 by x and then sin θ is square root of 1 minus cos square θ

So this gives this and this is equal to root 3 by 2 x plus one by two square root of one minus one by x square

So this is our one by y now if you see the option we are given one by y is equal to 2 times root 3 by x minus square root of 1 minus 1 by x square that is not correct

So what we get is that statement one is true but statement two is false

So b is the correct choice

So now let us move to question number five y, x be a solution of the differential equation one plus e to the x y prime plus y e to the x equal to one if y zero is equal to 2 then a y at minus 4 is equal to 0 b y at minus 2 is equal to 0 . c y, x has a critical point in the interval minus 1 comma 0 and d y, x has no critical point n minus one comma zero

So first we will solve this differential equation

So 1 plus e to the x y prime plus y e to the x equal to 1 the derivative d by dx of 1 plus e to the x times y this gives 1 plus e to the x times y prime plus y times derivative of 1 plus e to the x is e to the x and this implies 1 plus e to the x times y equal to x plus c of course this you could have also seen that this is a linear first order ode then you could find the integrating factor and do it but sometime it is easier if you can just realize this as the total derivative of some function

So this implies y is equal to x plus c divided by 1 plus e to the x given y 0 equal to 2

So this implies 2 is equal to c divided by 2 this implies c is equal to 4 therefore y is equal to x plus 4 divided by 1 plus e to the x

So by putting x equal to minus 4 this implies y at minus 4 is equal to 0 and y at minus 2 will be equal to 2 by 1 plus e to the minus 2 which is strictly greater than 0

So therefore option a is correct and b is wrong

So we got a is correct and b is wrong now c and d are asking whether y, x has any critical point in the interval minus one to zero

So remember that critical point is a point where the derivative of the function is 0

So y, x is equal to x plus 4 divided by 1 plus e to the x this implies y dash x is equal to 1 plus e to the x times the derivative of x plus 4 is 1 minus x plus 4 times derivative of the denominator is e to the x divided by 1 plus e to the x square

So y prime x is given by this expression this is equal to 0 at any point in the interval minus 1 to 0 or not

So let us calculate what is y prime at 0 if i put x equal to 0 then 1 plus e to the 0 is 1

So this is 2 minus 4 e to the 0 is 1 by 2 square

So that is minus 2 by 4 minus half which is less than 0 and y prime at minus 1 is equal to 1 plus e to the minus 1 minus 3 e to the minus 1 divided by 1 plus e to the minus 1 square

So we just have to see what is the sign of this this is equal to 1 minus 2 by e divided by 1 plus inverse square which is e minus 2 by e times one plus one by e square now we

know that e is bigger than two

So this is greater than zero

So y' at -1 is bigger than 0 and y' at 0 is less than 0 by the intermediate value theorem y' is equal to 0 for some x belonging to -1 to 0 which means that y has a critical point in the interval -1 to 0

So option c is correct and d is wrong

So a and c are the correct options okay let us move to question number six suppose we are given $f(x+y) = f(x)f'(y) + f'(x)f(y)$ for all x, y in \mathbb{R} and $f(0) = 1$ then find the value of $\log f(4)$

So we are given that f is a differentiable function which satisfies $f(x+y) = f(x)f'(y) + f'(x)f(y)$ for all x, y in the real line and also we are given the value of $f(0)$

So if we put $x = 0, y = 0$ then we see that we can calculate the value of $f'(0)$ this implies left hand side is $f(0) = f(0)f'(0) + f'(0)f(0)$ that is $2f(0)f'(0)$ which implies $f'(0) = 1$

So $f'(0)$ has to be equal to $1/2$.

So this is one fact that we derive now we have to calculate the value of $\log f(4)$

So we will try to find what is $f(x)$

So again putting $y = 0$ in the given equation we get $f(x+y) = f(x)f'(0) + f'(x)f(0)$. we all have already calculated the value of $f'(0)$

So this is equal to $f(x) \cdot \frac{1}{2} + f'(x) \cdot 1$ $f'(x) = \frac{1}{2}f(x)$ and this implies $f'(x) = \frac{1}{2}f(x)$ by integrating we get $\log f(x) = \frac{x}{2} + c$ $f(0) = 1$ implies $c = 0$

So therefore $\log f(x) = \frac{x}{2}$ we have to calculate $\log f(4)$

So this will be equal to 2 which is equal to 2

So let's check that this $f(x)$ satisfies the given functional equation

So we have $\log f(x) = \frac{x}{2}$ which implies $f(x) = e^{x/2}$ now if $f(x) = e^{x/2}$ then what is $f(x+y)$ this will be $e^{(x+y)/2}$ also what is $f(x)f'(y) + f'(x)f(y)$ this is equal to $f(x) \cdot \frac{1}{2}e^{y/2} + \frac{1}{2}e^{x/2} \cdot f(y)$ which is equal to $\frac{1}{2}e^{x/2}e^{y/2} + \frac{1}{2}e^{x/2}e^{y/2} = e^{(x+y)/2}$

So therefore this is true let us do one more problem the differential equation $dy/dx = \sqrt{1-y^2}/y$ determines a family of circles with option a is variable radii and a fixed center at the point $(0, 1)$ option b is variable radii and a fixed center at $(0, -1)$ c is fixed radius 1 and variable centers along the x axis and d is fixed radius 1 and variable centers along the y axis

So we know that the differential equation when we solve will get a family of curves and we have to choose the option among the given options

So we have $dy/dx = \sqrt{1-y^2}/y$ this is variable separable d

So we can write this as $y/\sqrt{1-y^2} dy = dx$ and this is easy to integrate we put $1-y^2 = u$ which implies $-2y dy = du$

So $y dy = -du/2$

So integral of $y/\sqrt{1-y^2} dy = \int -du/2\sqrt{u}$ divided by $\sqrt{1-y^2}$ is u

So this is $-\sqrt{1-y^2} + c$ which means $-\sqrt{1-y^2} + c = x$

So we get $x = -\sqrt{1-y^2} + c$

So this can be written as $\sqrt{1-y^2} = c - x$ which implies $1-y^2 = (c-x)^2$ or I can write this as $x^2 - 2cx + c^2 = 1-y^2$ also this implies $x^2 - 2cx + c^2 + y^2 = 1$ where c is an arbitrary constant

So this we see that this gives a circle with center at $(c, 0)$ and radius 1

So the radius for all of these circles is fixed 1 and center is $(c, 0)$ this lies on the x axis

So therefore option c is the only correct option here a, b and d are wrong all right

So this finishes lecture six on integral calculus thank you very much you