hello viewers welcome to iit palm mathematics channel this is lecture five on integral calculus today we are going to discuss some problems on differential equations

So let us start with the first some methods for solving first order ordinary differential equations

So what is a first order ordinary differential equation this in explicit form this is an equation of the form $\frac{dy}{dx}$ equal to f of x y where x is the independent variable and y is dependent on x

So $\frac{dy}{dx}$ is the derivative of y with respect to x is equal to some given function of x f is a given function of two variables

So this is explicit form sometime the equation is given in implicit form

So in implicit form a first order ordinary differential equation is any equation of the form some function capital f of x y and $\frac{dy}{dx}$ equal to zero where f is a given function of three variables

So let me just remind you some methods of solving first order ods solving first order ods

So the first simplest method is the method of variable separable

So in this method suppose the given ordinary differential equation can be written in the form f x dx equal to some g of y dy then we can simply integrate both sides

So we will write integral of f x dx equal to integral of g y dy this gives solutions in implicit form if possible we will write y as a function of x

So this is the first method where you can separate the x and y variables like this second method is what is called homogeneous

So this is an ode of the form $\frac{dy}{dx}$ equal to f of y by x

So if i can write the equation in this form where the derivative $\frac{dy}{dx}$ is some function of y by x then in such cases we can solve this by substituting y by x equal to a new variable u that is y is equal to u x

So if y is equal to u x then what is $\frac{dy}{dx}$ $\frac{dy}{dx}$ will be equal to u plus x $\frac{du}{dx}$ therefore we get u plus x $\frac{du}{dx}$ is equal to f of y by x is u f of u now you can see that this becomes a variable separable ode in the variables x and u is same as x $\frac{du}{dx}$ is equal to f of u minus u or $\frac{du}{f \text{ of } u \text{ minus } u}$ is equal to $\frac{dx}{x}$ now integrate both sides and finally we will put u equal to y by x to get the solution

So the third method that you need to know is how to solve linear first order odes

So a linear first order ode is of the form $\frac{dy}{dx}$ plus p x times y equal to g of x where p x and g x are given functions of x only

So in this case to solve this what we do is if we multiply this equation

So if we multiply the given equation by e to the power of integral of p x dx then the left hand side becomes

So the left hand side will be e to the power integral p x dx times $\frac{dy}{dx}$ plus p x times e to the power integral p x dx times y and now you can see that this is nothing but the derivative $\frac{d}{dx}$ of e to the power integral p x dx times y because by product rule if you differentiate this product you will get the exponential of integral p x dx times $\frac{dy}{dx}$ plus y times the derivative of e to the integral p x dx will give you e to the integral p x dx times the derivative of the exponent is simply p x

So now this equation

So the ode becomes $\frac{d}{dx}$ of e to the power integral p x dx times y is equal to e to the integral p x dx times g of x

So now we can simply integrate this this implies e to the integral p x dx times y is equal to integral of e to the integral p x dx times g x dx plus an arbitrary constant c thus we get this solution

So this factor that we multiplied e to the integral p x dx this is called integrating factors

So here e to the integral p x dx is called an integrating factor

So what we do first we calculate an integrating factor and then we find the solution

So let us now do some problems based on these methods

So let us start with the simple problem question one if y equal to y of x satisfies the differential equation eight root x times square root of 9 plus root x dy equal to square root of 4 plus square root nine plus root x this inverse dx for x greater than zero and y at 0 is equal to root 7 then find the value of y at two fifty six

So here if i write dy equal to 1 by square root of 4 plus square root 9 plus root x times 1 by 8 root x times square root of 9 plus root x dx

So we see that this is of the first kind variable separable equation now we just integrate both sides

So this will give y is equal to the integral of this now how to integrate this suppose we put u equal to four plus square root of nine plus root x then d u is equal to the derivative of this will give one by two square root of nine plus root x times the derivative of nine plus root x is one by two root x d x

So we see that we get d u equal to this

So now integrating this we get y is equal to integral of 1 by square root of u times d u by two and this is equal to square root of u plus c

So y is equal to putting again u equal to this we get 4 plus square root of 9 plus root x plus c y at 0 equal to root 7 we get c is equal to zero

So therefore y at x is equal to square root of four plus square root 9 plus root x and now we can easily calculate y at 256 will be equal to square root of 4 plus square root 9 plus square root of 256 will give 16 and then this becomes square root of 4 plus 9 plus 16 is 25 square root gives 5 and this gives a square root of 9 which is 3

So that is the answer let's do second problem let f from r to r be a differentiable function such that f of zero equal to zero if y equal to f x satisfies d y d x equal to 2 plus 5 y times 5 y minus 2 then find the value of limit x approaching negative infinity of f of x

So first we will try to solve this ode

So we have d y d x equal to two plus five y times five y minus two

So this can be rewritten as one by five y plus two times five y minus two d y equal to d x

So again this is variable separable equation now we integrate both sides now this can be written in partial fractions this implies we can write this as 1 by 5 y minus 2 minus one by five y plus two if we do then we get in the numerator four

So one by four times this is equal to the integrand here

So this d y equal to integral d x

So this gives one by four the integral of one by five y minus 2 will give 1 by 5 times natural log of mod 5 y minus 2 minus this is 1 by 5 natural log of mod five y plus two is equal to x plus c

So if we multiply by twenty this gives natural log of mod five y minus two by five y plus two is equal to twenty x plus twenty c now we are given that y zero is equal to zero this will imply natural log of if we put y equal to zero zero minus two by zero plus two mod equal to x is zero

So zero plus twenty c this implies c is equal to zero therefore log of mod phi y minus two by five y plus two this is equal to twenty x which implies mod of five y minus two by five y plus two is e to the power twenty x now suppose l is equal to limit x tends to minus infinity of f of x then if we take the limit as x tends to minus infinity and we get five l minus two by five l plus two this mod is equal to limit x tending to minus infinity of e to the power twenty x and this limit is equal to zero

So this implies five l minus two equal to zero which implies l is equal to two by five

So the limit extending to minus infinity of f of x is equal to two by five lets do problem number three

So it is given that a curve passes through the point one comma pi by six and let the slope of the tangent to the curve at any point x comma y b y by x plus secant of y by x for x bigger than zero then the equation of the curve is you are given four options a is sine of y by x equal to natural log of x plus half b cos x y by x equal to log x plus half c secant of 2 y by x equal to log x plus 2 and d is cos 2y by x equal to log x plus half

So we know that the slope of tangent at a point x comma y on the curve is given by d y d x

So we are given d y d x is equal to y by x plus secant of y by x for x bigger than zero

So we see here that d y d x is given as a function of y by x

So this is a homogeneous form

So we put y equal to u x and then we know that this d y d x will be u plus x d u d x equal to the right hand side y by x is u plus secant of u

So u cancels and this implies x d u d x is equal to secant u which implies cos u d u equal to d x by x by integrating this we get sine u equal to log of x plus c x is given to be positive

So we do not need to put log mod x and that is sine of y by x is equal to log x plus c now to find the value of c we have to use the condition that when x is equal to one y is y is equal to pi by six because one comma pi by six is given to lie on the curve

So therefore if we put y equal to pi by 6 and x equal to 1 we get sine pi by 6 equal to log 1 plus c log 1 is 0 and sin pi by 6 is half

So this implies c is equal to half therefore sine y by x is equal to log x plus half

So this says that option a is the correct answer and b c and d are wrong

So this was an example of an ode which is in the homogeneous form let's do question number four let f from zero infinity to r be a differentiable function such that f prime x is equal to 2 minus f x by x and f one is not equal to one then which of the following options is or are correct the first option is limit extending to 0 plus f prime 1 by x equal to 1 b is limit extending to 0 plus x times f 1 by x equal to 2 c is limit x tending to 0 plus x square times f prime x equal to 0 d is mod fx is less than equal to two for all x belonging to open interval zero two

So what we are given is an ode we write y equal to f x then we have f prime x is d y d x is equal to 2 minus y by x and this can be rewritten as d y d x plus 1 by x times y equal to 2 this is linear ode we know how to solve in fact in this case if you multiply this by x this gives x times d y d x plus y equal to 2 x and now you can clearly see that left hand side is nothing but the derivative d by d x of x times y equal to 2 x then by integrating this we get x times y equal to x square plus c which implies y is equal to x square plus c divided by x

So this can be written as x plus c by x

So that is f of x is given by x plus c by x now we are given one more condition that f of one is not equal to one this implies f of one is equal to one plus c and since f of one is not equal to one we get c is not equal to zero

So therefore we have f of x is x plus c by x for some c not equal to 0. now let us look at the options

So option a says the limit of f prime of one by x as x approaches zero

So let us calculate

So what is f prime of x f prime x is equal to 1 minus c by x square which implies f prime of 1 by x is equal to 1 minus c x square now as x approaches 0 this approaches 1 as x approaches zero plus

So therefore option a is correct

So option a is correct option b is saying limit of x times f of 1 by x

So let us calculate what is x times f of 1 by x this is equal to x times f of 1 by x will be equal to 1 by x plus c x which is equal to 1 plus c x square this approaches 1 as x approaches 0 plus

So option b is saying that this limit is equal to 2

So this is wrong therefore b is wrong option c is asking for x square times f prime x limit as x approaches 0 plus

So we have already calculated f prime x

So x square times f prime x this is equal to x square times f prime x is 1 minus c x square which is equal to x square minus c

So therefore limit of x approaching 0 plus x square times f prime x is equal to minus c and we know that c is non-zero

So minus c is not equal to 0 but option c is saying that this limit is equal to 0

So this is wrong

So here note that f prime x was already known to be 2 minus f x by x

So one may try to multiply this by x square and then try to find the limit note that since f prime x is equal to two minus f x by x x square times f prime x will be equal to 2 x square minus x times f x now here you can make mistake and think that as x approaches 0 this 2 x square approaches 0 and then we have x times f of x

So of course x approaches 0 and then you might think that x times f x also approaches zero and

So mistake thinking that limit x approaching zero plus of x f x equal to zero this need not be true because this limit of f x as x approaches 0 might be infinity or minus infinity

So limit f x as x approaches 0 plus this need not be finite

So if you thought that this limit is equal to 0 then you will think that option c is correct but it is not correct

So now lets look at option d

So option d is saying that mod f x is less than equal to two for x between zero and two what is f x f x is x plus c by x

So f x is equal to x plus c by x and c is given to be non-zero since c by x this

approaches infinity as x approaches 0 plus therefore f x is not bounded on the interval zero two

So therefore option d is also wrong

So we get d is also wrong

So only option a is correct option here let's do question number five let y prime x plus y x times g prime x is equal to g x times g prime x or x in r and y zero is equal to zero where g x is a given non constant differentiable function on r with g zero equal to g of two equal to zero then the value of y of two is what

So what we have is d y d x plus g prime x times y is equal to g x times g prime x

So this is again you see that this is linear

So what we do is we find first integrating factor e to the power integral p x d x p x is g prime x d x

So this is equal to e to the g x

So we get e to the g x is an integrating factor here and then we multiply by e to the g x and we get y times integrating factor e to the g x equal to integral of integrating factor e to the g x times the right hand side is g x times g prime x d x now to integrate this we use integration by parts integration by parts integral of we will write this as g x times e to the g x g prime x d x this is equal to integral of g x times d e to the power g x and then by integration by parts this is equal to g x times e to the power g x minus integral of g prime x times e to the g x d x which is equal to g x e to the power g x minus this is nothing but derivative of e to the power g x

So this gives e to the g x plus c

So therefore we have y times e to the g x is equal to g x minus one times e to the g x plus c which implies y is equal to g x minus 1 plus c e to the minus g x now using y at 0 equal to 0 this implies 0 equal to g of 0 minus 1 plus c times e to the minus g of 0 g of 0 is given to be equal to 0

So this is equal to 0 minus 1 plus c which implies c is equal to 1.

So therefore y is equal to g x minus 1 plus e to the minus g x now we need to calculate y at 2

So y of 2 will be equal to g of two minus one plus e to the power minus g of two g of two is given to be zero this is zero minus one plus e to the zero

So this is equal to minus one plus one which is zero

So therefore y two is equal to zero

So again this problem was a first order linear ode

So we can easily solve this let us do question number six let f from zero infinity to r be a continuous function such that f x is equal to 1 minus 2 x plus integral 0 to x e to the power x minus t times f of t dt for all x belonging to zero infinity then we are given four options the curve y equal to f x passes through the point one comma two b is the curve y equal to f x passes through the point two comma minus one is the area of the region r equal to x y belonging to zero one cross r such that f x is less than equal to y is less than equal to square root of one minus x square is pi minus two by four and option d is the area of r is pi minus one by four

So in this case we are given a function f which satisfies this given equation here if we see we do not have a differential equation

So what we are given is f x is equal to 1 minus 2 x plus integral 0 to x e to the power x minus t f t d t

So first by putting x equal to 0 we get f of 0 equal to 1 because this integral is integral from zero to zero which is zero

So f zero equal to one this is one thing we get and we can write this say f x as one minus two x plus you see this integral is with respect to t

So this e to the x comes out of this integral and then we have integral from zero to x of e to the minus t times f of t d t

So actually here what we have is called an integral equation because we have a function and integral but by differentiating this we can get a differential equation

So differentiating with respect to x we get f prime x is equal to minus 2 plus to differentiate this we will use product rule

So we have e to the x times 0 to x e to the minus t f t d t this is by differentiating the first term plus e to the x times the derivative of the second term is simply e to the minus x times f x

So this gives f prime x is equal to minus 2 plus this again i can write as 0 to x of e to the x minus t f t dt plus f of x now this integral is nothing but f x minus 1 plus 2 x

So this i will write as minus 2 plus f x minus 1 plus 2 x and then we have plus f x

So this gives f prime x minus two f x is equal to two x minus three now this is a linear ode

So what is integrating factor here integrating factor is e to the power integral of minus two d x which is e to the minus two x

So by multiplying by e to the minus two x we get d d x of e to the minus two x times f x is equal to e to the minus 2 x times 2 x minus 3 which implies e to the minus 2 x f x is equal to integral of 2 x minus 3 times e to the minus 2 x d x this again we integrate by parts

So this is equal to 2 x minus 3 times e to the minus 2 x by minus 2 minus integral of derivative of 2x minus 3 gives 2 times e to the minus two x by minus two d x

So this is equal to minus half two x minus three e to the minus two x this becomes plus e to the minus two x by minus two plus c multiplying by e to the two x this gives f x is equal to minus half two x minus three minus half plus c e to the power 2 x

So therefore f x is equal to c e to the power 2 x minus x and then we have plus three by two minus half plus one now we have calculated f of 0 is equal to 1

So this implies 1 is equal to c plus 1 which implies c is equal to 0.

So therefore f x is equal to 1 minus x

So we get f x is simply 1 minus x now let us look at the options

So the first option says that the curve y equal to f x passes through the point one comma two

So what is f of one f of one will be one minus one equal to zero

So the curve passes through the point one comma zero not one comma two

So option a is wrong option b is passes through the point 2 comma minus 1

So what is f of 2 is 1 minus 2 equal to minus 1 therefore b is correct

So option b is correct option c and d are asking to find area of this region

So what is the region is y is between f x and square root of one minus x square

So this region r is x y now into 0 1 cross r such that f x is 1 minus x is less than equal to y is less than equal to square root of one minus x square we need to find what is area of r

So if we look at this region if you look at y equal to 1 minus x is this a straight line which passes to the point one comma zero and zero comma one and y equal to square root of 1 minus x square for x between 0 and 1 is this circular arc this is y equal to square root of one minus x square means x square plus y square equal to one

So this is the circular arc of radius one now now this region is for y region is bounded from a verb by the curve y equal to square root of 1 minus x square and from below by 1 minus x

So the region is this region

So here you do not even have to integrate to find this area because this area of r is nothing but the area of this quarter circle minus area of this triangle

So this is 1 4 times area of circle of radius 1 is pi 1 square minus area of this right angle triangle is half times 1 times 1

So this is pi by 4 minus half which is equal to pi minus two by four

So option c is correct d is wrong therefore c is correct d is wrong

So this problem shows that sometime the equation is not given in terms of derivative rather it is given in terms of integral

So it is an integral equation but by differentiating this we can convert this into differential equation and you have to find some initial condition here by putting x equal to 0 we got f of 0 equal to 1 and then you get a first order differential equation with some initial condition and then you can solve that to solve this given integral equation all right

So this finishes lecture five on integral calculus in the next lecture we will do some more problems on differential equations thank you you