hello viewers welcome to iit palm mathematics channel this is lecture 4 on integral calculus

So we will continue doing some more problems let us start with problem number one this says that let s be the area of the region bounded by y equal to e to the power minus x square y equal to 0 x equal to 0 and x equal to 1 then which of the following options correct is s is bigger than equal to 1 by e b is s is greater than equal to 1 minus 1 by e c is s is less than equal to 1 by 4 times 1 plus 1 by square root of e and d is s is less than equal to 1 by root 2 plus 1 by square root of e times 1 minus 1 by root 2. let us try to solve this problem

So lets first draw the region we have x y axis and the graph of y equal to e to the minus x square this looks like this and at x equal to 0 the value is 1 and it keeps on decreasing as x increases

So this region is the area bounded by this curve y equal to 0 is the x axis and x equal to zero is the y axis and x equal to one

So this is the region area of this region is given to be equal to s now if we see what is this value

So at x equal to 1 e to the minus x square will give 1 by e

So this point is 1 comma 1 by e

So clearly we can see that this area is bigger than equal to area of this rectangle which is equal to 1 by e times this is 1

So so area s is bigger than equal to area of rectangle let's call this point a b c o a b c which is equal to 1 times 1 by e

So s is bigger than equal to one by e is clearly correct

So a option is correct now b says s is bigger than equal to one minus one by e

So if we see e to the minus x square if we compare e to the minus x square with e to the minus x then we see that e to the minus x square is bigger than equal to e to the minus x for all x between zero to one this is because x square is less than equal to x for x in 0 1.

So e to the minus x square will be bigger than equal to e to the minus x

So therefore the integral of 0 to 1 e to the minus x square dx this is bigger than equal to integral of 0 to 1 e to the minus x dx and this integral is nothing but 1 minus 1 by e

So therefore option b is also correct

So this option a and option b are both correct now we have to see option c and option d

So here we have to find whether s is less than equal to something

So again we should try to see if we can see this to be less than equal to some area if we see option d we have s is less than equal to 1 by root 2 plus 1 by square root of e times 1 minus 1 by root 2.

So if we draw this this is one zero if we look at the point one by root two then one by root two is the area of this rectangle is 1 by root 2 times height is 1 and what about this rectangle

So this point the value at 1 by root 2 will be 1 by root 2 comma y is e to the minus x square

So that will be e to the minus half

So this height is equal to one by square root of e

So by the figure we see that s is less than equal to sum of area of these two rectangles

So area of the first rectangle is 1 by root 2 plus the area of the second rectangle is height is 1 by square root of e times 1 minus 1 by root 2.

So that is our option d

So a b and d are correct let us try to see if c is correct or not

So c option is saying that s is less than equal to 1 by 4 times 1 plus 1 by square root of e we have already seen that s is greater than equal to 1 minus 1 by e

So let's try to see whether this quantity is less than or bigger than one minus one by e

So if we see one minus one by e minus 1 by 4 times 1 plus 1 by square root of e this is in option c

So this is equal to 1 minus 1 by 4 that is 3 by 4 minus 1 by e minus 1 by 4 square root of e which is equal to 1 by 4 e times three e minus four minus square root of e now we know that e is is between two and three is approximately two point seven one

So we can use this to see that 3e minus 4 minus square root of e this will be bigger than 3 times 2 minus 4 minus square root of 3

So this is equal to 2 minus square root of 3 which is clearly bigger than 0

So therefore we see that this difference is bigger than 0

So therefore 1 minus 1 by e is bigger than 1 by 4 times 1 plus 1 by square root of e since s is bigger than equal to 1 minus 1 by e will have s is bigger than 1 by 4 times 1 plus 1 by square root of e

So therefore option c is wrong

So this says that options a b and d are correct and c is wrong

So here you should note that this integral e to the minus x square dx from 0 to 1 it is not possible to evaluate it exactly

So therefore you have to use these inequalities you cannot calculate the exact value of this integral now let us move to second problem

So suppose f is given to be a function from the closed interval half one to zero infinity let this be a non constant differentiable function such that f prime of x is strictly less than 2 times f x and f at half is equal to 1 then the value of integral of f x d x from half to one lies in the interval option a is 2 e minus 1 to 2 e b is e minus one to two e minus one c is e minus one by two to e minus one and d option is 0 to e minus 1 by 2. let us try to solve this problem

So what we are given is f prime x is less than 2 times f of x f prime x is less than 2 times f x this implies f prime x minus two f x is strictly less than zero this is for all x in half to one now what we can do is we multiply this by e to the power minus two x

So e to the minus two x times f prime x minus two f x we know that exponential is always positive

So this will also be less than zero why we multiplied by e to the minus two x because by doing this we see that this can be written as d by d x of e to the minus two x times f x because by product rule this is e to the minus two x times derivative of f is f prime x plus derivative of e to the minus two x will give minus two e to the minus two x times f x

So that gives you this

So this is less than zero for all x and half to one now we know that if the derivative of function is negative in an interval then this implies that e to the minus 2 x times f x is a decreasing function in the interval half to one

So therefore e to the minus two x f x will be strictly less than e to the power minus two times half times f at half for all x greater than half

So f of half is given to be equal to one

So this is equal to 1 by e which implies f of x is less than e to the power two x minus one for x in half to one if f x is less than a function g of x then this will imply integral half to one f x d x is less than integral of half to one e to the two x minus one d x and this is equal to e to the two x minus one by two from half to one which is equal to at x equal to one this will give e by two and at x equal to half it is e to the zero is one by two

So this is e minus one by two

So therefore this integral half to one f x d x this is strictly less than e minus 1 by 2 also since f x is greater than zero integral half to one f x d x this will be bigger than zero

So therefore this lies between zero and e minus one by two

So this option d is correct and all the other options you can see are wrong because this integral is less than e minus 1 by 2

So option c is wrong similarly option b is wrong and option a is on

So note that in the solution we don't need to use that f is a non-constant function but this is given in the problem because if it was not specified that f is non constant then you can take f 2 with a constant function because f of half is given to be equal to 1 we can take f x equal to one for all x in this interval that satisfies f prime x will be zero which is less than two times f x and in that case this integral because f x is equal to one this is simply equal to half and half clearly lies between zero and e minus one by two

So one can easily get that option d is correct but if we are given that f is non constant function then we cannot be sure that this is correct unless we solve this let us move to question number three let f x and g x be non constant differentiable functions on r such that f prime of x is equal to e to the power f x minus g x times g prime x for all x in r and f of 1 equal to g of 2 equal to 1 then a f of 2 is less than 1 minus natural log of 2 b is f of 2 is bigger than 1 minus log 2 c is g of 1 is bigger than 1 minus log 2 and d is g of one is less than one minus log two

So we are given f prime x is equal to e to the power f x minus g x times g prime x

So this implies e to the minus f x times f prime x is equal to e to the minus g x times g prime x but here this is nothing but derivative of e to the power minus f x and this is derivative of e to the minus g x

So what we have is derivative of this is equal to derivative of this for all x this implies e to the minus f x is equal to e to the minus g x plus some constant for some constant c now we are given the value of f of 1 is 1 and g of 2 is 1.

So we will use those

So therefore e to the power minus f of one will be equal to e to the minus g of one plus c f of one is given to be equal to one

So this implies e to the minus g of 1 plus c is equal to e to the minus 1 and e to the minus f of 2 is equal to e to the minus g of 2 plus c g of 2 is 1

So this is e to the minus 1 plus c

So we get this these two equations we do not know the value of c

So we can eliminate c from these two equations

So from one and two we have c is e to the minus f of 2 minus 1 by e from 2 this is equal to one by e minus e to the minus g of one

So this implies e to the power minus f of two plus e to the power minus g of one is equal to two by e now we know that exponential this is always positive

So this implies e to the minus f of 2 is strictly less than 2 by e and e to the minus g of one is also strictly less than two by e

So this implies e to the f of 2 is bigger than e by 2 and e to the g of 1 is bigger than e by 2 then by taking log f of 2 will be bigger than log e by two which is log e is one minus log two and similarly g of one is bigger than one minus log two

So option b f of two bigger than one minus log two and g of one bigger than one minus log two these are correct and a and d are wrong again you should note that if in this question it was not given that these are non constant functions then you can take f and g to be the constant function one and then clearly this equality is satisfied because both sides are zero

So in that case f of two will also be equal to one and g of one will be one and which is clearly bigger than one minus log two

So you can easily get option b and c without doing any work let us now do problem number four we are given g of a equal to integral from zero to one t to the power minus a times 1 minus t to the power a minus 1 dt for a belonging to open interval 0 to 1 also it is given that g is differentiable on the open interval 0 1 then find the values of g at half and derivative g prime at half

So first let us try to find g of half by putting a equal to half g of half will be integral of 0 to 1 t to the power minus half times 1 minus t to the power a minus 1 is again minus half dt this is equal to integral from 0 to 1 1 by square root of t times 1 minus t dt now this can be written as integral from 0 to 1 1 by square root of we have t minus t square inside square root

So this can be written as 1 by 4 minus t minus half square d t 1 by 4 will cancel and will get minus t square plus t now this we know is nothing but sine inverse of t minus half divided by half evaluated between zero and one

So for t equal to one this will be sine inverse of half by half

So sine inverse of 1 minus sine inverse of at 0 it is minus 1

So this is equal to pi by 2 minus minus pi by 2 that's equal to pi

So therefore g at half is equal to pi next we have to calculate the value g prime at half

So there are two ways of doing one is we can try to differentiate this g of a and then put a equal to half and another way is to notice that let us calculate what is g of 1 minus a g of a is given to be equal to integral from 0 to 1 t to the minus a 1 minus t to the power a minus 1 dt

So therefore g of 1 minus a is equal to 0 to 1 t to the power minus of 1 minus a and 1 minus t to the power a is replaced by 1 minus a minus 1 dt

So this is equal to integral from 0 to 1 t to the power a minus 1 and 1 minus t to the power minus a dt now this this is same as integral of 1 minus t to the power a minus 1 and t to the power minus a dt from 0 to 1 this is because integral of a to b f x d x is same as integral of a to b f of a plus b minus x d x

So now this is equal to g of a

So therefore we have got that g of a is equal to g of 1 minus a for all a in 0 to 1 by differentiating we get g prime of a is equal to minus g prime at one minus a now we can put a equal to half we get g prime of half is equal to minus g prime of half this implies

g prime half must be zero

So the value of g prime half is equal to zero another way to calculate this g prime of half is we have g of a is equal to integral 0 to 1 t to the minus a 1 minus t to the a minus 1 dt this implies g prime of a will be equal to integral of 0 to 1 the partial derivative with respect to a of t to the minus a 1 minus t to the a minus 1 dt now we can calculate this derivative with respect to a of the integrand that is equal to minus t to the minus a log t 1 minus t to the a minus 1 plus t to the minus a 1 minus t to the a minus 1 and log 1 minus t dt and now if we put a equal to half we see that this is equal to zero at a equal to half therefore g prime at half is equal to zero ok lets do question number five

So we have to find the value of integral from 0 to 1 4 x cube times the second derivative of 1 minus x square raise to power 5. d x

So one direct way let us first calculate this second derivative and then integrate but if you see this will involve lot of computations

So this will be a lengthy process because we have to take derivative two times of this then multiply by four x cube then integrate it

So that will be a huge polynomial and this will take too much time

So a smarter way is to use integration by parts formula

So let us recall integration by parts formula

So we have if we write integral of f x times g prime x dx this is equal to fx times gx minus integral of f prime x g of x dx or there is another way if we have two functions f x and g x multiplied together then this integral will be f x times you integrate the second function g x dx minus integral of you differentiate the first function f prime x and integrate the second function g x dx and then you integrate it

So we will use this formula

So let us write f x is equal to 1 minus x square to the 5 then we have to calculate integral of 0 to 1 of 4 x cube times f double prime x dx

So by integration by parts formula this is equal to 4 x cube times if we see f double prime x is derivative of f prime x

So this is equal to f prime x from 0 to 1 minus integral of 0 to 1 the derivative of 4 x cube will give 12 x square times f prime x dx now this we can again use integration by part

So this is 4 x cube f prime x from 0 to 1 minus this will be 12 x square times f x from 0 to 1 minus integral 0 to 1 of 12 x square when we differentiate we get 24 x times f x d x now since f x is equal to one minus x square to the five what is f prime at one this will be equal to 0 because you can differentiate this and then put x equal to 1 you have 1 minus x square factor

So that will be equal to 0 or you can notice that x equal to 1 is a repeated root of this function f of x and therefore f prime 1 must be equal to 0.

So therefore this part is equal to 0 at one and at x equal to zero we have four x cube term

So this is zero similarly f of one is also zero

So this this part is also equal to zero because f of one is zero and at zero we have x square term

So therefore this thing is nothing but zero to one four x cube times f double prime x dx is equal to 24 times integral 0 to 1 x times fx dx

So this is equal to twenty four zero to one x f x is one minus x square to the power five d x now this can be easily integrated

So this is equal to if we put 1 minus x square equal to y then we get minus two x d x is d y then this say is nothing but twelve times integral zero to one y to the five d y which is equal to 2 y to the 6 from 0 to 1 and this is equal to 2

So this can be easily integrated

So therefore the answer is the value of the integral is equal to two let's do question number six if i sub n denotes this definite integral from minus pi to pi of sine of nx divided by 1 plus pi to the power x times sine x dx for n equal to 0 1 2 and

So on for every non negative integer we have defined i n to be this integral then which of the following options are correct is i n equal to i n plus 2 for all n b summation of i 2 m plus 1 for m equal to 1 to 10 this is equal to 10 pi c summation i 2 m for m equal to 1 to 10 equal to 0 and option d is i n equal to i n plus 1 for all n

So we are given this integral i n this is integral from minus pi to pi of for this thing

So we know that this will be equal to integral of 0 to pi of sine nx by 1 plus pi to the

x times sine x this is f of x plus f of minus x that will be sine of minus n x by 1 plus pi to the minus x times sine of minus x d x this is because integral minus a to a of f x d x is nothing but integral 0 to a of f of x plus f of minus x d x now if we add these two you see that i n is nothing but integral 0 to pi of sine nx by sine x dx other thing adds to 1

So we get this now clearly we can calculate if we put n equal to 0 i 0 is equal to 0 because sine 0 is 0 and also i 1 we can easily calculate i 1 will be integral 0 to pi sine x by sine x dx this is simply equal to pi

So clearly this shows that i n is not equal to i n plus 1 for n equal to 0

So option d is wrong i zero is not equal to i one implies d is wrong now let us say compare i n plus 2 with i n

So if we look at i n plus 2 minus i n this will be integral 0 to pi for n plus 2 this will be sine of n plus 2 x minus sine n x divided by sine x d x now we can use the formula sin c minus sin d

So this is equal to integral of 0 to pi 2 cos c plus d by two

So that is n plus two plus n is two n plus two divided by two

So that is n plus one x times sine c minus d by 2 n plus 2 x minus n x will give sine 2 x by 2 will give sine x divided by sine x dx

So sine x cancels and this is simply two times sine n plus one x divided by n plus one between zero and pi which is equal to zero because sine at any integer multiple of pi is zero

So therefore i n plus two is equal to i n for all n

So that is option a is correct now we also get b and c

So since we have already seen that i zero is zero we have i two m equal to zero for all m and i one is equal to pi

So i 2 m plus 1 is equal to pi for all m

So therefore if we see option b summation of i two m plus one for m equal to one to ten each i two m plus one is pi

So this will give ten pi

So this is correct and i two m is zero for all m

So therefore the sum is also zero

So a b and c are correct

So let us do one more problem question number seven the total number of distinct x in 0 1 for which integral of 0 to x t square by 1 plus t to the 4 dt is equal to 2x minus 1 is what

So we have to find the number of x for which this integral is equal to two x minus one

So we let us write f x equal to integral of 0 to x t square by 1 plus t to the 4 d t minus two x minus one for x belonging to zero one

So clearly f is continuous and differentiable function also lets see what is the value of f of 0 if we put x equal to zero the integral is zero to zero

So this is zero minus this will be minus one

So this is equal to one

So f of zero is positive and if we put x equal to 1 then f of 1 is equal to integral 0 to 1 t square by one plus t to the four d t minus two x minus one at x equal to one is one

So we have this integral minus now for t between 0 and 1 t square by 1 plus t to the 4 is strictly less than one

So therefore integral zero to one t square by one plus t to the four d t this will be less than 1 and therefore f of 1 is the difference is negative

So we have f is a continuous function f at zero is positive and f at one is negative

So by the intermediate value theorem by intermediate value theorem for continuous function we know that there exists at least one x in the interval 0 to 1 for which f x is equal to 0

So we know that the number of x for which this integral equal to 2 x minus one is at least one let us see if we can have more than one now what is f prime of x

So f x is integral of this minus 2 x minus 1

So if we differentiate this f prime x will be x square by 1 plus x to the 4 minus 2. now we know that x square by 1 plus x to the 4 is strictly less than 1

So this is less than 1 minus 2 which is minus 1

So f prime x is less than 0 this implies f is strictly decreasing in the interval 0 to 1.

So f is strictly decreasing function which means that f cannot have more than one zeros

So this implies f can have at most one zero in zero one

So we got that f has at least one zero in this interval and this says that f has at most one zero
So therefore the number of x for which f x equal to zero is one
So that's the answer to this problem
So this finishes lecture four on integral calculus in the next lecture we will do some more problems thank you you