

hello viewers welcome to iit pal mathematics channel

So this is lecture 3 on integral calculus

So let us do some problems based on integration

So first let us recall some facts about area enclosed by curves suppose we have a function $f(x)$ is greater than equal to zero

So we have the graph of this function $y = f(x)$ which lies above the x axis and we have this line $x = a$ and $x = b$

So the area bounded by the curve $y = f(x)$ the lines $x = a$ $x = b$ and the x axis is given by the definite integral of a to b of $f(x) dx$ right

So this definite integral gives the area under the curve $y = f(x)$ from $x = a$ to $x = b$ now suppose we have two functions the area between the curves $y = f(x)$ $y = g(x)$ and $x = a$ and $x = b$ is given by

So suppose we have two functions $f(x)$ and another function $g(x)$ and we need to find this area

So in this figure $f(x)$ is greater than $g(x)$ this will be integral from a to b of $f(x) - g(x) dx$ $f(x)$ is greater than equal to $g(x)$ this is because the integral of a to b $f(x) dx$ gives you the area between the curve $y = f(x)$ and x axis from $x = a$ to b and the other one integral of $g(x) dx$ from a to b gives the area under this one

So the difference gives the area in general this area will be equal to integral a to b of $\max(f(x) - g(x), 0) dx$

So we have to look at the upper curve and lower curve and take the difference and then integrate to find the area

So now let us look at some problems first one let us find the area enclosed by the curves $y = \sin(x) + \cos(x)$ and $y = \max(\cos(x) - \sin(x), 0)$ over the interval 0 to $\pi/2$

So let $f(x)$ denote the first function $\sin(x) + \cos(x)$ and $g(x)$ is $\max(\cos(x) - \sin(x), 0)$ for x belonging to 0 to $\pi/2$

So in the interval 0 to $\pi/2$ both $\sin(x)$ and $\cos(x)$ are non negative

So here note that this $g(x)$ is $\max(\cos(x) - \sin(x), 0)$ this is less than equal to $\max(\cos(x) + \sin(x), 0)$ this is by the triangle inequality and because x is in 0 to $\pi/2$ this is equal to $\cos(x) + \sin(x)$ for x in 0 to $\pi/2$ but $\cos(x) + \sin(x)$ is same as $f(x)$

So we have in this case $f(x)$ is greater than equal to $g(x)$ for x in 0 to $\pi/2$

So therefore the required area is equal to integral of $f(x) - g(x) dx$ from 0 to $\pi/2$.

So this is equal to integral 0 to $\pi/2$ $f(x) - g(x) dx$ $f(x)$ is $\sin(x) + \cos(x)$ $g(x)$ is $\max(\cos(x) - \sin(x), 0)$ now we have $\max(\cos(x) - \sin(x), 0)$ this is equal to $\cos(x) - \sin(x)$ if $\cos(x)$ is greater than equal to $\sin(x)$ and that is if x is between 0 to $\pi/4$ we know that $\cos(x)$ is bigger than equal to $\sin(x)$ and for x between $\pi/4$ to $\pi/2$ $\sin(x)$ is bigger than $\cos(x)$

So this will be equal to $\sin(x) - \cos(x)$

So therefore this integral we can write as integral from 0 to $\pi/4$ and then we have $\sin(x) + \cos(x) - (\cos(x) - \sin(x)) dx$ plus $\pi/4$ to $\pi/2$ $\sin(x) + \cos(x) - (\sin(x) - \cos(x)) dx$

So this is equal to the first one $\cos(x)$ cancels and we have 0 to $\pi/4$ $2 \sin(x) dx$ plus 0 to $\pi/4$ and $\pi/4$ to $\pi/2$ $2 \cos(x) dx$ and then this is simply $2 \int_0^{\pi/4} \sin(x) dx + 2 \int_{\pi/4}^{\pi/2} \cos(x) dx$ and then this is equal to $2[-\cos(x)]_0^{\pi/4} + 2[\sin(x)]_{\pi/4}^{\pi/2}$ which is equal to $2(1 - \frac{1}{\sqrt{2}}) + 2(1 - \frac{1}{\sqrt{2}})$ is $4 - 2\sqrt{2}$.

So this is equal to $4 - 2\sqrt{2}$ or $4 - 2\sqrt{2}$

So this is the required area

So of course in this question you could have drawn the graph of these functions $\sin(x) + \cos(x)$ and $\max(\cos(x) - \sin(x), 0)$ and then find the area but its not even necessary to draw the graph because we can see that in this case $f(x)$ is greater than equal to $g(x)$ now let us move to question number two again we will find an area

So find the area of the region in the first quadrant given by $xy = 8$ is that x times y is less than equal to eight and one is less than equal to y less than equal to x^2

So let us first draw this region

So we have $xy = 8$ is the rectangular hyperbola and $y = x^2$ is the parabola like this

So first let us see where these two intersect

So we have this is $x y$ equal to eight and this is y equal to x square

So if i put y equal to x square then i get x times x square that is x cube equal to eight

So x is equal to two

So this point is two comma four and another curve is y equal to one

So y equal to 1 is this straight line

So now let us figure out the region

So again this y equal to one intersects the parabola y equal to x square at the point one comma one and it intersects the rectangular hyperbola $x y$ equal to eight at y is one

So x will be eight

So this is eight comma one now what is the region the region is given in the first quadrant by $x y$ less than equal to eight let me draw this $x y$ less than equal to eight in this yellow this is the reason $x y$ less than equal to eight is below this curve y equal to eight by x and y has to be bigger than equal to one and less than equal to x square

So this this yellow region is $x y$ less than equal to eight and in the first quadrant now let us draw the region y greater than equal to one is above this line this is the region y is bigger than equal to one in first quadrant and the reason y is less than equal to x square is this region in yellow in orange because this is below the curve y equal to x square

So what we have to find is the intersection of these reasons

So that is nothing but this region in red

So this is the area that we have to calculate now once we have drawn this this is now easy to do

So here what is the upper curve let me write this upper curve $f(x)$ this is equal to the upper curve is x square if one is less than equal to x and less than equal to two from one to two if you see if this is x equal to one this is two and this is x equal to eight

So from one to two it is y equal to x square and from two to eight it is y equal to eight by x and the lower curve $g(x)$ is simply y equal to one for x between one to eight

So what we have to find the area is integral from one to eight of $f(x) - g(x) dx$ which is equal to integral from one to two of $x^2 - 1 dx$ plus two to eight of $\frac{8}{x} - 1 dx$ this can be easily evaluated this is equal to $\frac{x^3}{3} - x$ from 1 to 2 and plus $8 \log x - x$ from 2 to 8 this is equal to $\frac{8}{3} \log 8 - 8 + 8 \log 2 - 2 + 2 - 1$ which is equal to $\frac{8}{3} \log 4$ is same as $16 \log 2 - 14$ by 3.

So this is the answer for this now here note that because this upper curve was different from in different interval from one to two and two to eight we had to calculate this area by splitting it into sum of two integrals this problems can also be done in another way

So lets write another way let me draw this region again we have y equal to x square we have this $x y$ equal to eight and y equal to one this point is two comma four this is 1 comma 1 and this is 8 comma 1. now here you can see that this region is bounded between these two lines y equal to one and y equal to four and the left curve here is this one y equal to x square and the right curve is y equal to eight by x

So this area i can write as instead of integrating with respect to x if i do with respect to y and this is the integral from one to four of $8/y - y^2 dy$ means x is equal to \sqrt{y} the advantage here is that we can evaluate this as just one integral

So this is equal to $8 \log y - \frac{y^3}{3}$ from one to four which is equal to $8 \log 4 - \frac{64}{3} + \frac{1}{3}$ which is equal to $16 \log 2 - \frac{63}{3}$ which is $16 \log 2 - 21$ by 3

So this is eight minus one

So sixteen log two minus fourteen by three

So in some problems instead of writing as integral of $f(x) dx$ we can do integral of some $f(y) dy$ and that might be easier to evaluate let us move to question number three here is given that the line x equal to α divides the area of the region r equal to $x y$ in r two says that y is between x cube and x and x is between zero and one

So the line x equal to α divides this region into equal parts then which of the following options are correct the first option is α is greater than 0 and less than equal to half second is α is greater than half and less than 1 c is 2 times α to the 4 minus $4 \alpha^2$ plus 1 equal to 0 and d option is α to the 4 plus $4 \alpha^2$ minus 1 equal to 0.

So first we should see what is the region r region r is bounded by the curve y equal to x cube and y equal to x for x between zero and one

So we have y equal to x and y equal to x cube looks like this this is x equal to zero and x equal to one

So this this is the region r first let us calculate what is the area of the region r area of the region r is equal to integral from 0 to 1 of the upper curve is x and the lower curve is x cube

So x minus x cube dx which is equal to x square by two minus x to the four by four between zero and one which is equal to one by two minus one by four this is equal to one by four

So that is the total area of this region r now we have to find this line x equal to α which divides this in equal parts that means that the integral

So α is says that the integral of 0 to α of x minus x cube $d x$ this should be equal to half of the area of region r which is one by four

So this is equal to one by eight now this will give α square by 2 minus α to the 4 by 4 is equal to 1 by 8 which implies if i multiply by 8 and 4 α square minus 2 α to the 4 is equal to 1 which means 2 α to the 4 minus 4 α square plus 1 equal to 0 .

So if we see the options this is same as option c

So option c is correct also immediately now we can rule out option d because if we see that α satisfies this option c 2 α to the 4 minus four α square plus one equal to zero and if it also satisfies α to the four plus four α square minus one equal to zero then if we add these two then we get three α to the four is equal to zero which will give α equal to zero but clearly α equal to zero does not satisfy this option c

So this d option is wrong now what about option a and b

So one way is to this equation is quadratic in α square

So we can solve for α square from this equation and hence get the value of α another simpler way is it we have to see whether α is less than equal to half or α is bigger than half

So what we can do is simply calculate this area for α equal to half

So if we calculate zero to half of x minus x cube $d x$ this will be equal to just put α equal to half

So this will be one by eight minus one by 2 to the 4 by 4

So this is 1 by 8 minus 1 by 64 which is 7 by 64 which is less than half

So if we integrate from 0 to half we get the area which is less than half

So therefore α must be bigger than half

So this implies α has to be between half and one

So that means that option b is correct and a is wrong therefore option b is correct of course we can calculate the value of α directly we have 2 α to the 4 minus 4 α square plus 1 equal to 0 this will imply α square is equal to 4 plus minus square root of 16 minus 8 by 4 this is equal to 1 plus minus 1 by root 2. now since α has to be less than 1 we have α square equal to 1 minus 1 by root 2 and this will imply α is square root of one minus one by root two and you can see that this is bigger than half but we did not have to actually calculate this value of α lets move to next problem question four let f v a function from minus one to to zero infinity b a continuous function such that f of x is equal to f of 1 minus x for all x in minus 1 to 2 and let r_1 equal to integral of minus one to two of x times $f x d x$ and r_2 be the area of the region bounded by y equal to $f x$ x equal to minus one x equal to two and the x axis then we have four options given first is r_1 equal to two r_2 b is r_1 equal to three r_2 c is two r_1 equal to r_2 and option d is three r_1 equal to r_2

So let us write what is r_1 r_1 is given by integral from minus one to two of x times $f x d x$ now we will use the fact that it is given that f of x is same as f of one minus x

So this is equal to minus one to two of x times f of one minus x dx now putting 1 minus x equal to y where this integral minus 1 to 2 of x times f of 1 minus x dx is equal to integral from minus 1 to 2 of 1 minus y times f of $y d y$

So this is equal to integral from minus one to two of $f y d y$ minus integral minus one to two of $y f y d y$

So therefore this r_1 is equal to the first integral minus one to two $f y d y$ is nothing but r_2 and this second integral is again r_1 this implies two r_1 is equal to r_2

So we have option c is correct us move to question number five find the area of the region x, y in r^2 says that y is bigger than equal to square root of x plus three and five y is less than equal to x plus nine is less than equal to fifteen

So let us try to draw this region

So first we have this five y less than equal to x plus nine less than equal to fifteen this is equivalent to five y is less than equal to x plus nine and x is less than equal to six

So we have y is less than equal to x plus nine by five and x is less than equal to six

So this region we have x equal to 6 and y equal to x plus 9 by 5 is the straight line this intersects x equal to six if we put x equal to six then y is six plus nine fifteen by five is three

So this is the point six comma three and when x is 0 it is 9 by 5 and y becomes 0 when x is equal to minus 9.

So this is minus nine comma zero this is the straight line y equal to x plus nine by five

So y is less than equal to x plus 9 by 5 is the region below this line and x less than equal to 6 is the reason to the left of the line x equal to six

So this gives this region which is bounded between this also we have to see this restriction that y is greater than equal to square root of x plus three

So this part is the region five y less than equal to x plus nine less than equal to fifteen now let us look at y equal to square root of x plus three this is equal to square root of x plus three if x plus three is non negative that is x is greater than equal to minus 3 and it is square root of minus of x plus 3 if x is less than minus 3.

So this this is of course symmetric about x equal to minus three

So we can try to draw this for x bigger than equal to minus three

So we have this is x equal to minus three and y equal to square root of x plus three this is this part of the parabola like this this is y equal to square root of x plus three for x bigger than equal to minus three and this $f(x)$ is symmetric about this

So this will be the other side it is like this

So this is the graph of $f(x)$ this part is y equal to square root of minus x plus 3. now what we have is the region is y is bigger than equal to $f(x)$

So y bigger than equal to $f(x)$ will be the region above this this is the region y bigger than equal to square root of x plus three lets also see where this curve intersects the line

So y equal to square root of x plus three intersects the line x equal to 6 at if we put x equal to six then y is equal to three

So at six comma three which note is also the intersection of these two straight lines and the line y equal to x plus nine by five at the points minus four comma one and one comma two this is by equating this is square root of x plus three equal to x plus nine by five and then you can square and you can find this point of intersection

So let us draw this the region is as below if this is minus three comma zero this is the intersection six comma three this is the line y equal to x plus 9 by 5 this is x equal to 6 and this is y equal to $f(x)$

So the region is this region that we have to find the area and these points are minus 4 comma one and another point of intersection is here which is one comma two this part is y equal to square root of x plus 3 and this is y equal to square root of minus x plus three

So now we can find this area by integrating this upper curve which is in this case the straight line minus the lower curve which is this part of these two parabolas

So to find this area note that here the upper one because its an its a straight line we can find this lets call these points at a b c and d

So this is area of trapezium abcd minus the area of these two

So minus what is the area of this left part will be integral from minus 4 to minus 3 of square root of minus x plus 3 dx and then we have to subtract minus integral from minus three to one of square root of x plus three dx let us evaluate this integral

So integral minus three to one square root of x plus three this will be equal to x plus 3 raised to power 3 by 2 times 2 by 3 minus 3 to 1 which is 2 by 3 we put x equal to 1 this is 4 to the 3 by 2 which is 8 minus 0 this is 16 by 3 and integral of minus 4 to minus 3 of square root of minus x minus 3 dx this will be equal to minus x minus 3 raised to power 3 by 2 then we have minus 2 by 3 from minus 4 to minus three this is minus two by three times at x equal to minus three this is zero minus when we put x equal to minus four this will be equal to one

So this is equal to two by three and area of trapezium a b c d is equal to half times

this will be $a d + b c$ times $c d$

So this is equal to half $a d$ is equal to $1 b c$ is $2 \cdot 1 + 2$ and $c d$ is equal to 1 minus 4

So that is five

So this is equal to fifteen by two

So therefore area of the region is equal to fifteen by two minus sixteen by three plus two by three that is fifteen by two minus eighteen by three is six that gives three by two

So this is the answer let me do one more problem let $f(x)$ equal to $\int_0^x (x^2 + \pi/6 - 2 \cos^2 t) dt$ for every real number x and f from zero half to zero infinity be a continuous function for each a in zero to half if $f'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$ then find the value of $f(0)$.

So what is given to us that this $f'(a) + 2$ this is the area of the region bounded by this curve $y = f(x)$ dx from 0 to a what is given to us is $f'(x)$ is equal to $\int_0^x (x^2 + \pi/6 - 2 \cos^2 t) dt$ note that we only need f'

So we should not try to evaluate this integral

So this implies $f'(x)$ we know that to find the derivative of this integral from two functions of x this will be given by $2 \cos^2$ of the upper limit is $x^2 + \pi/6$ times the derivative of $x^2 + \pi/6$ is $2x$ minus $2 \cos^2$ of the lower limit x times the derivative of the lower limit which is one

So this is equal to $4x \cos^2 x^2 + \pi/6 - 2 \cos^2 x$ therefore $f'(a) + 2$ which will be equal to $4a \cos^2 a^2 + \pi/6 - 2 \cos^2 a + 2$ which I can write as $4a \cos^2 a^2 + \pi/6 + 2 \sin^2 a$ because $\sin^2 a = 1 - \cos^2 a$ now this is given to be equal to integral

So therefore integral from 0 to a of $f(x) dx$ is equal to $4a \cos^2 a^2 + \pi/6 + 2 \sin^2 a$ and we have to find the value of $f(0)$

So we can differentiate this differentiating with respect to a we get $f'(a)$ is equal to derivative will be $4 \cos^2 a^2 + \pi/6 + 4a \cos^2 a^2 + \pi/6 + 4a \sin^2 a^2 + \pi/6 + 4a \cos^2 a^2 + \pi/6 + 4a \sin^2 a^2 + \pi/6$ derivative of $2 \sin^2 a$ will be $4 \sin a \cos a$ now put $a = 0$

So $f(0)$ is equal to $4 \cos^2 \pi/6 + \pi/6 + 2 \sin^2 0$ plus we have a times this

So this will be 0 and \sin of 0 is 0 \cos of $\pi/6$ is $\sqrt{3}/2$ $\sqrt{3}/2$ square

So this is equal to 3 therefore $f(0)$ is equal to 3 . ok

So this finishes lecture 3 on integral calculus in the next lecture we will do some more problems thank you