

hello viewers welcome to lecture two on integral calculus in the first lecture we discussed few problems on definite integral in this lecture we will do some more problems So first i will start with a few problems on definite integrals as the limit of sum So let me first explain the concepts of definite integrals as limits of sums So suppose we have a function f of x and we want to find the definite integral of f of x from a to b

So what we know is that the definite integral gives the area under the graph of this function y equal to $f(x)$ from a to b now to evaluate this area what we do is we sub divide this interval a to b into n equal parts

So let us call x_0 is a , x_1 is $a + h$, x_2 is $a + 2h$ and So on and x_n is equal to $a + nh$ this is my x_0 and i have x_1 , x_2 and the last one is x_n

So then what is h ? h is x_n minus x_0 and this x_n is equal to b

So this is equal to $b - a$

So therefore h is $b - a$ over n

So this interval of length $b - a$ we are dividing into n equal parts

So each is of length $(b - a) / n$ now we can draw these rectangles and suppose i find the areas of these rectangles whose width is h and height is equal to this $f(x_i)$

So so what we have is this integral of a to b $f(x) dx$ is equal to limit as n tends to infinity of sum of these areas of the rectangles

So what is area of rectangle this will be h times $f(a + kh)$ where k is varying from 1 to n this is same as limit n tends to infinity of summation k equal to one to n $(b - a) / n$ times $f(a + k(b - a) / n)$

So this is the formula for definite integral in terms of limit of some of these areas

So in particular if a is 0 and b is one then we get integral zero to one $f(x) dx$ is equal to limit as n tends to infinity of summation k equal to one to n $1/n$ times $f(k/n)$

So one thing you should note that here what we have done is in each sub interval we have taken the value of the function at the right end point and then we got these areas and we have added we can also do instead of the right end point we can take the value at the left end point and do

So we also have if we take the values of the function at the left end points of each sub interval instead of these right end points we still get integral a to b $f(x) dx$

So we can write integral of a to b $f(x) dx$ is equal to limit n tends to infinity in summation k equal to zero to $n - 1$ and the width is $(b - a) / n$ times $f(a + k(b - a) / n)$

So instead of starting from k equal to one to n if we start from k equal to zero to $n - 1$ this gives the integral

So in some problems maybe you have to take this in fact one can take any point in the sub interval and that will still give the definite integral of $f(x)$ from a to b

So now lets start with some problems

So question one suppose for a in \mathbb{R} with $\text{mod } a$ strictly bigger than one let limit of n tends to infinity $1 + \sqrt[3]{2} + \sqrt[3]{3}$

So on up to $\sqrt[3]{n}$ divided by n to the power $7/3$ times $1 + a^n + 1 + a^n + 2 + a^{2n}$ this limit is given to be equal to 54 then the possible values of a is or r we are given four options a is minus nine or minus six c is seven and d option is eight

So if you see here we have limit of the ratio of two sums

So we will try to write this as definite integral

So what we do is we have limit n tends to infinity in the numerator we have summation of r to the power one by three r equal to one to n and in the denominator i have n to the seven by three times summation one by $a^n + r^2$ r equal to 1 to n now somehow and this this limit is we have to write as definite integral

So what i will do is i will write this is equal to limit n tends to infinity i write this as summation of r by n raise to power one by three then i have to multiply by n to the one by three

So this is the numerator and in denominator i have n to the seven by three and again i want to write as r by n

So i will take n common from the denominator here

So i have summation of r equal to one to n $1 + r$ by n square and then i will have n square in the denominator i have $1 + n$ square now if you see this n to the 1 by 3

is in the numerator and i have n to the 7 by three divided by n square that is again n to the one by three

So this cancels and i have limit n tends to infinity of summation r by n to the one by three divided by limit n tends to infinity summation are equal to one to n of one by a plus r by n square now this can easily be written as the first one is the integral from zero to one of x to the one by three dx because if we take $f(x)$ equal to x to the one by three and then use this formula $\int_0^1 f(x) dx$ is limit of $\frac{1}{n}$ times summation $f(k)$ by n we get this

So this i can also write as limit of n tends to infinity of one by n times summation r by n to the one by three and the denominator is again $\frac{1}{n}$ times summation 1 by a plus r by n square

So now the numerator becomes the integral of x to the one by three from zero to one and the denominator is integral from zero to one of one by a plus x square this can easily be evaluated

So this will give three by four x to the four by three from zero to one divided by minus one by a plus x from zero to one

So this is equal to three by four a times a plus one simplification now this limit was given to be equal to fifty four

So three by four a times a plus one is equal to fifty four this implies a square plus a is equal to 72 which implies a square plus a minus 72 is equal to 0 and this gives a minus eight times a plus nine equal to zero

So a is either eight or minus nine

So among the options given we see that a equal to minus nine and eight are possible but minus six and seven these are not possible

So this finishes problem one us move to question number 2.

So question 2 says for every natural number n we have y_n equal to $\frac{1}{n}$ times n plus 1 times n plus 2 product of these up to n plus n whole raise to power one by n and if limit of y_n as n tends to infinity is equal to 1 then the value of greatest integer of 1 equal to what

So we are given y_n is equal to one by n times n plus one n plus two up to n plus n whole raised to power one by n

So if i take this n inside then this will be equal to n plus one by n times n plus two by n up to n plus n by n raised to power one by n which is same as one plus one by n into one plus two by n up to one plus n by n raise to power one by n

So here instead of sum this is product of some terms

So naturally we can take natural log

So this implies natural log of y_n will be equal to $\frac{1}{n}$ times summation of log of 1 plus k by n k is equal to 1 to n

So now we can see that we have got this is equal to $\frac{1}{n}$ times summation of some f of k by n

So this implies limit of n tends to infinity of log y_n is equal to limit n tends to infinity of summation of $\frac{1}{n}$ times log 1 plus k by n which is equal to integral of 0 to 1 of log of 1 plus x dx this we can easily evaluate

So we can integrate by parts here this is equal to x times log one plus x min from zero to one minus integral zero to one of x by one plus x dx this is integrating by parts

So this is equal to at x equal to one we get log two minus at x equal to zero this is zero minus this integral again i can do easily this is equal to $\frac{1}{1+x}$ dx

So this is equal to log 2 minus x minus log of 1 plus x from 0 to 1 this gives log 2 minus 1 minus log 2 and at 0 it is 0.

So this gives 2 natural log of 2 minus 1 which is equal to natural log of 4 minus 1 i can write as natural log of e

So this is equal to log of 4 by e

So what we have got is that the limit of natural log of y_n is equal to this

So limit of natural log of y_n is equal to natural log of four by e taking exponential we get limit of y_n is equal to four by e and this was denoted by l

So l is equal to four by e what we have to find is the greatest integer of l

So since we know that e is between two and three e is approximately two point seven one So it is between two and three

So four by e will be between four by three and four by two which is two

So it is strictly between one and two this implies greatest integer of l is equal to one

So this is the answer to this second problem let us move to question number three

integral because the second integral is not easy to evaluate

So now we should look at the options and try to see what we can deduce

So so if we see options a and b we have to compare f of half with f of one and f of one by three with f of two by three

So so if we can calculate the derivative we can see whether the function is increasing or decreasing and then we can deduce this

So that is easy to do here

So we differentiate this differentiating with respect to x we get

So \log of $f(x)$ if we differentiate we will get $f'(x)$ by $f(x)$ equal to this is integral $\int_0^x \frac{1}{1+y^2} dy$ if we differentiate by fundamental theorem of calculus this will simply give \log of $1+x$ by $1+x^2$ and we always had x bigger than 0

So we get this as $f'(x)$ by $f(x)$

So this implies $f'(x)$ is equal to $f(x)$ times \log of $1+x$ by $1+x^2$ note that $f(x)$ is clearly positive because $f(x)$ was $\int_0^x \frac{1}{1+y^2} dy$ if x is positive then this term is positive for every x

So $f(x)$ is positive and what about \log of $1+x$ by $1+x^2$

So and \log of $1+x$ by $1+x^2$ this is positive if x is between 0 and 1 because if x is less than 1 then $1+x$ will be bigger than $1+x^2$ for x between 0 and 1 x^2 is smaller than x

So this ratio $1+x$ by $1+x^2$ this is bigger than one and \log of anything bigger than one is positive but if x is bigger than 1 then the denominator has x^2 this is bigger than x

So this ratio $1+x$ by $1+x^2$ becomes less than 1

So this becomes less than zero if x is greater than one

So therefore $f'(x)$ is greater than zero if x is between zero and 1 and this is less than 0 if x is bigger than 1 .

So what does this tell us

So this implies f is an increasing function on the interval 0 to 1 and it is decreasing function on one to infinity

So if we see the options f of half and f of one between zero and one f is increasing

So f of half will be less than f of one this implies f of half has to be less than f of one and f of one by three will be less than f of two by three

So option a is incorrect but b is correct now c and d options c asks for $f'(2)$ and d is comparing $f'(3)$ by $f(3)$ with $f'(2)$ by $f(2)$.

So these again we can get from what we have calculated

So let's see what is $f'(2)$ $f'(2)$ will be equal to $f(2)$ times \log of $1+2$ by $1+2^2$ this is equal to $f(2)$ times \log of 3 by 5 .

So $f(2)$ is positive and \log of 3 by 5 is negative

So this will be less than 0

So option c says $f'(2) \leq 0$ we know that $f'(2)$ is less than 0

So c option is correct and what about d option we have to see $f'(3)$ by $f(3)$ and $f'(2)$ by $f(2)$

So we we have $f'(x)$ by $f(x)$ is equal to \log of $1+x$ by $1+x^2$ therefore $f'(3)$ by $f(3)$ is equal to \log of 4 by 10 which is \log of 2 by 5 . $f'(2)$ by $f(2)$ this we have already calculated is \log of 3 by 5 .

So \log is strictly increasing function

So \log of 2 by 5 is less than \log of 3 by 5

So this implies $f'(3)$ by $f(3)$ is strictly less than $f'(2)$ by $f(2)$

So our option d that means is wrong we have the other inequality

So b and c are correct option ok lets move to question number four let $f(x)$ is equal to $\int_1^x \frac{e^{-t}}{t} dt$ for x in zero to infinity then which of the following options are correct f is increasing on one infinity b f is decreasing on the interval 0 to 1 c $f(x) + f(1/x)$ this is equal to 0 for all x and 0 to infinity and d option is $f(2)^x$ is an odd function of x on \mathbb{R} here we are given $f(x)$ as a definite integral and then we have to find these options

So so because we are being asked for increasing and decreasing we can calculate the derivative of this function to see whether f is increasing or decreasing in some interval

So let us recall if we take the derivative of some $f(t)$ from some function a to b then what is this

So this is equal to we evaluate f at the upper end point b of x and then multiply by the

derivative $b^x - f'(a) \cdot x$ times a^x right

So this is the general formula for derivative of definite integral where the limits of the integrals are functions of x in particular when we have integral from a to x then we simply get $f(x)$ but here f we have to evaluate at the upper end point times the derivative of the upper one minus f at the lower end point times the derivative of that

So we have $f(x) = 1/x^2 = e^{-2 \ln x}$ by $t = \ln x$ divided by $t dt$

So if we differentiate this then $f'(x)$ will be equal to $e^{-2 \ln x}$ first we put t equal to x in the integrand

So that is $x + 1/x$ times the derivative of x which is 1 minus we have to put t equal to $1/x$

So $e^{-2 \ln x} = 1/x^2$ by t will become x divided by t is $1/x$ times the derivative of $1/x$ will give $-1/x^2$

So this is nothing but $2/x^3$ times $e^{-2 \ln x} = 1/x^2$ divided by x and because exponential is always positive this is greater than 0 for all x bigger than zero

So this implies that f is strictly increasing on the interval zero to infinity

So f is an increasing function on the whole zero to infinity

So therefore option a is correct and b is wrong note that we could have also deduced the same thing without calculating the derivative of this function because

So let me write another way we see that the interval $1/x$ to x this increases as x increases why is this because as x increases the lower end point $1/x$ will decrease and the upper end point x increases

So therefore as x increases the interval becomes bigger and bigger also the integrand is positive because integrand is exponential divided by t for t positive this is always positive

So if we have a positive function then if we integrate it on a bigger interval this will be bigger

So therefore $f(x)$ is an increasing function right

So even without calculating the derivative one can easily see this by just seeing that the interval on which we are integrating this positive function is becoming bigger as x increases now to see option c and d we have to look at $f(x) + f(1/x)$

So let's see what is $f(x) + f(1/x)$ will be integral from x to $1/x$ of e^{-t} plus $e^{-1/t}$ divided by t dt we put $t = 1/y$ then $dt = -1/y^2 dy$ and when $t = x$ $y = 1/x$ and then $t = 1/x$ $y = x$ So this $f(x) + f(1/x)$ will be equal to integral from $1/x$ to x of e^{-t} is $1/y$ by y

So $1/y + 1/t$ will be y divided by t is $1/y$ and dt is $-1/y^2 dy$ this is same as $1/x^2$ we have a minus sign and then we have e^{-y} plus $1/y$ divided by $y dy$ which is same thing as $-f'(x)$

So this implies $f(x) + f(1/x)$ this is always zero

So $f(x) + f(1/x) = 0$ this is option c is correct now what about option d this follows from option c

So now also if we write $g(x) = f(2^x)$ then $g'(x) = f'(2^x) \cdot \ln 2$ which is equal to $f'(1/2^x)$ and we know that $f'(1/x) = -f'(x)$

So this is $-f'(2^x)$ which is equal to $-g'(x)$

So this implies $g(x)$ is an odd function

So that gives option d is also correct

So option a c and d are correct options all right

So this finishes lecture two on integral calculus in the next lecture we will discuss some more problems thank you