

hello viewers welcome to iit pal mathematics channel

So this is part of a series of problem solving sessions i will be giving a few problem solving sessions on integral calculus and differential equations

So this is lecture one of integral calculus

So let us start by doing some problems the problems are mainly chosen from previous years j advanced papers and i will review the important concepts used in the problems along with the problems

So lets start with problem number one

So question one says that if i is equal to 2 over π integral from $-\pi$ by 4 to π by 4 of dx over $1 + e^{\sin x}$ times $2 - \cos x$ then twenty seven times i square equal to what

So basically we have to evaluate this definite integral and then calculate the value of twenty seven i square

So if we see here we have integral of the form $\int_{-a}^a f(x) dx$

So let me recall

So for any function $f(x)$ integral of $\int_{-a}^a f(x) dx$ this is equal to integral of $\int_0^a f(x) dx + \int_0^a f(-x) dx$

So instead of evaluating from $-a$ to a we can just evaluate from 0 to a

So the proof of this is very simple

So what we do is integral from $-a$ to a of $f(x) dx$ this can be written as integral from $-a$ to 0 of $f(x) dx$ plus integral from 0 to a of $f(x) dx$ this is a simple properties of definite integral that if we have integral from a to b then we can split into sum of two integrals from a to c and c to b now what we do is in the first integral putting x equal to $-y$ in the first integral

So if we put x equal to $-y$ dx will become $-dy$ and when x equal to $-a$ y is equal to a and when x is equal to 0 y is zero

So integral of $\int_{-a}^0 f(x) dx$ can be written as integral from a to 0 of $f(-y) dy$ and dx is $-dy$ which is same thing as minus of integral a to 0 of $f(-y) dy$ and this can be written as integral from 0 to a of $f(-y) dy$ which can also be written as integral from 0 to a of $f(-x) dx$ therefore integral from $-a$ to a of $f(x) dx$ is equal to integral of $\int_0^a f(-x) dx + \int_0^a f(x) dx$ which is this our formula of course there are two special cases which are important in many problems

So if f is an odd function that is $f(-x) = -f(x)$ then then this $\int_{-a}^a f(x) dx$ will become zero

So therefore integral of $\int_{-a}^a f(x) dx$ will be equal to zero and the second one is if f is an even function that is $f(-x) = f(x)$ for all x then integral of $\int_{-a}^a f(x) dx$ is same as two times integral from 0 to a of $f(x) dx$ now now let us do the solution to the problem

So we have $f(x) = \frac{1}{1 + e^{\sin x}}$ times $2 - \cos 2x$ now let us evaluate $f(-x)$ this is equal to $\frac{1}{1 + e^{\sin(-x)}}$ times $2 - \cos 2(-x)$ now $\sin(-x) = -\sin x$

So this will be equal to $\frac{1}{1 + e^{-\sin x}}$ and \cos is an even function So this is same as $\frac{1}{1 + e^{-\sin x}}$ times $2 - \cos 2x$ this can be simplified as $e^{\sin x}$ divided by $1 + e^{\sin x}$ times $2 - \cos 2x$

So we see that the denominator is same as that of $f(x)$

So now if i add $f(x) + f(-x)$ $f(x) + f(-x)$ is equal to $\frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{-\sin x}}$ times $2 - \cos 2x$

So $1 + e^{\sin x}$ will cancel and this is equal to $\frac{1}{2 - \cos 2x}$ now i will use the formula for $\cos 2x = 2 \cos^2 x - 1$

So this is equal to $\frac{1}{3 - 2 \cos^2 x}$

So what we have is the integral i is equal to we are 2 by π factor that will be 0 to π by 4 of $\frac{1}{3 - 2 \cos^2 x} dx$ now this integral is not difficult to do what we can do is we write this integrand as $\frac{\sec^2 x}{3 \sec^2 x - 2}$ and now it should be obvious to you that if we substitute $u = \tan x$ we can write this as 0 to π by 4 $\frac{1}{3 - 2(1 + \tan^2 x)}$ dx

So now put $u = \tan x$ then $du = \sec^2 x dx$ and the integration limit when x is zero u is equal to $\tan 0$ which is 0 and when x is equal to π by 4 u is equal to 1

So i is equal to 2 over π integral from 0 to 1 of $\frac{1}{3 - 2(1 + u^2)}$ du over this was $3 - 2 \tan^2 x$

So $3u^2 + 1$. now this is in standard integral

So this can be written if i take three common from the denominator i get two over three $\pi \int_0^1 \frac{du}{u^2 + 1}$ by three which i will write as one by root three square

So this is equal to $\frac{2}{3\pi}$ and the integral of $\frac{1}{u^2 + a^2}$ is $\frac{1}{a}$

So $\frac{1}{\sqrt{3}}$ times $\tan^{-1} \frac{u}{\sqrt{3}}$ divided by $\frac{1}{\sqrt{3}}$. and this has to be evaluated between zero and one

So this is equal to $\frac{2}{\sqrt{3}\pi} [\tan^{-1} \frac{1}{\sqrt{3}} - \tan^{-1} 0]$ $\tan^{-1} \frac{1}{\sqrt{3}}$ is equal to $\frac{\pi}{6}$ and $\tan^{-1} 0$ is zero

So we get this

So i is equal to π cancels here two by three root three this implies i square will be equal to four by twenty seven which means twenty seven i square is equal to four

So this is the answer to the first problem

So the question number two is find the value of the integral i is equal to integral from 0 to $\frac{\pi}{2}$ of $\frac{3 \cos^2 \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$ d theta

So to solve this problem what we do is first we substitute $\theta = \frac{\pi}{2} - \phi$ then $d\theta = -d\phi$ and when $\theta = 0$ $\phi = \frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$ $\phi = 0$ also $\cos \theta = \sin \phi$ and $\sin \theta = \cos \phi$

So the integral will become integral from zero to $\frac{\pi}{2}$ of $\frac{3 \sin^2 \phi}{\sqrt{\sin^2 \phi + \cos^2 \phi}}$ d phi

So note that here i have skipped one step

So the integral will be from $\frac{\pi}{2}$ to zero and then will have minus d phi and then the minus of this integral can be written as integral from 0 to $\frac{\pi}{2}$ of this again this integral instead of phi i can use theta

So this can also be written as $\int_0^{\pi/2} \frac{3 \cos^2 \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$ d theta

So now adding this

So let let me call the original equation as equation one and this one is equation two

So adding 1 and 2 we get $2I = \int_0^{\pi/2} \frac{3 \cos^2 \theta + \sin^2 \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$ d theta

So now we can cancel this and this will be $\int_0^{\pi/2} \frac{3 \cos^2 \theta + \sin^2 \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$ d theta now the problem reduces to evaluating this integral now here again to integrate this one what we can do is if i take a square root of $\cos^2 \theta$ common from the denominator and this will become $\frac{3 \cos^2 \theta + \sin^2 \theta}{\cos^2 \theta \sqrt{1 + \tan^2 \theta}}$

So now we got $\tan \theta$ and if you see i can bring in the numerator $3 \sec^2 \theta$ by $1 + \tan^2 \theta$ raised to the power 4 d theta

So now we can put $\tan \theta = t$ then $\sec^2 \theta d\theta = dt$ and the limit when $\theta = 0$ $t = 0$ and when $\theta = \frac{\pi}{2}$ $t = \infty$

So this integral is equal to

So this was $2I = \int_0^{\infty} \frac{3 \cdot 2t dt}{1 + t^2}$ d t

So so 2 can be cancelled and this implies i is equal to $\frac{3}{2} \int_0^{\infty} \frac{t dt}{t^2 + 1}$ now this can be easily done by writing this as $\frac{t}{t^2 + 1} = \frac{1}{2} \frac{2t}{t^2 + 1}$

So this is equal to $\frac{3}{2} \int_0^{\infty} \frac{t dt}{t^2 + 1}$ which is equal to $\frac{3}{2} [\frac{1}{2} \ln |t^2 + 1|]_0^{\infty}$ which will give $\frac{3}{4} \ln |t^2 + 1|$ from zero to infinity now as t tends to infinity $\frac{1}{t^2 + 1}$ goes to zero and this also goes to zero

So this will be equal to $\frac{3}{2} \ln 2$ minus this will become at $t = 0$ this becomes $\frac{3}{4} \ln 1$ which is zero

So therefore the answer is the value of i is equal to $\frac{3}{4} \ln 2$

So let us go to problem number three

So again we will evaluate one definite integral the question is find the value of the integral $\int_0^{\frac{1}{2}} \frac{1 + \sqrt{3}}{x^2 + 1} dx$ let us try to solve this problem
So first we will do a little simplification and write this $\int_0^{\frac{1}{2}} \frac{1 + \sqrt{3}}{x^2 + 1} dx$ we have $\frac{1 + \sqrt{3}}{x^2 + 1}$
So that will become $\frac{1 + \sqrt{3}}{x^2 + 1}$ times $1 - x^2$ raised to power $\frac{1}{2}$ times $1 + x^2$ raised to power $\frac{1}{2}$ by dx
So that is $\frac{1 + \sqrt{3}}{2}$.

So note that in this range 0 to $\frac{1}{2}$ $1 + x$ and $1 - x$ are both positive

So now I will write this as $\frac{1 + \sqrt{3}}{2} \int_0^{\frac{1}{2}} \frac{1 - x^2}{1 + x^2} dx$ by I will write this $1 - x^2$ to the power $\frac{3}{2}$ as $(1 - x)^2 (1 + x)^{\frac{1}{2}}$

So we have $1 + x$ to the power $\frac{1}{2}$ and $1 - x$ to the power $\frac{1}{2}$

So that will become square root of $1 - x^2$

So we get dx by $1 - x^2$ times $1 - x^2$ square and a square root now here we have a simple substitution because we have this one minus x^2 term we should try substitution $x = \sin \theta$

So then dx will be equal to $\cos \theta d\theta$ and when x is 0 θ is equal to 0 and when x is equal to $\frac{1}{2}$ $\sin \theta = \frac{1}{2}$

So θ is $\frac{\pi}{6}$

So therefore this integral $\int_0^{\frac{1}{2}} \frac{1 + \sqrt{3}}{x^2 + 1} dx$ is equal to $\frac{1 + \sqrt{3}}{2} \int_0^{\frac{\pi}{6}} \frac{\cos \theta d\theta}{1 - \sin^2 \theta}$ and square root of $1 - x^2$ square is equal to $\cos \theta$

So $\cos \theta$ cancels and we get integral of $d\theta$ by $1 - \sin^2 \theta$ now this is straight forward what we do is you multiply and divide by $1 + \sin \theta$ then it becomes $\frac{1 + \sin \theta}{1 - \sin^2 \theta}$

So that is $\frac{1 + \sin \theta}{\cos^2 \theta}$ this is equal to $\frac{1 + \sin \theta}{\cos^2 \theta}$ times $\cos^2 \theta$ plus $\frac{\sin \theta}{\cos^2 \theta}$ is $\sec^2 \theta + \tan \theta$ now you must be knowing the integral of this
So this is equal to $\frac{1 + \sqrt{3}}{2} [\tan \theta + \sec \theta]$ evaluated between 0 to $\frac{\pi}{6}$ this is $\frac{1 + \sqrt{3}}{2} (1 + \sqrt{3} - 0 - 1)$ this is $\frac{1 + \sqrt{3}}{2} \times \sqrt{3}$

So $\frac{1 + \sqrt{3}}{2} \times \sqrt{3} - \frac{1 + \sqrt{3}}{2} \times 1$ which is equal to $\frac{3 - 1}{2}$ which is 1 .

So therefore the value of the integral is equal to 2

So let's move to next problem question number four

So the problem is that let f from r to r be a differentiable function such that $f(0) = 0$, $f(\frac{\pi}{2}) = 3$ and $f'(0) = 1$.

So we are given a differentiable function whose values at 0 and $\frac{\pi}{2}$ are given and $f'(0) = 1$ is given now if $g(x) = \int_x^{\frac{\pi}{2}} f'(t) \cos at - \cot t \cos at f(t) dt$ for x in 0 to $\frac{\pi}{2}$ open at 0 and close at $\frac{\pi}{2}$ then we have to find limit of $g(x)$ as x approaches 0 .

So this $g(x)$ is given in terms of this integral

So first we should try to evaluate this integral to find the formula for $g(x)$ and then we will try to find this limit

So note that here if you see the integrand it is $f'(t) \cos at - \cot t \cos at f(t)$ this is nothing but the derivative of $f(t) \cos at$ because if we use the product rule for derivative then this gives derivative of $f(t) \cos at$ and derivative of $\cos at$ is $-\sin at$ times $f(t)$

So therefore this $g(x)$ is equal to $\int_x^{\frac{\pi}{2}} d(f(t) \cos at)$

So once we know the anti-derivative of the integrand then by the fundamental theorem of calculus this is equal to $f(t) \cos at$ evaluated between x and $\frac{\pi}{2}$

So that is equal to $f(\frac{\pi}{2}) \cos a \frac{\pi}{2} - f(x) \cos ax$ now $f(\frac{\pi}{2})$ was given to us $f(\frac{\pi}{2}) = 3$

So this is equal to $3 \cos a \frac{\pi}{2} - f(x) \cos ax$

So this is $3 \cos a \frac{\pi}{2} - f(x) \cos ax$

So note that $\cos a \frac{\pi}{2} = 1$ by sine x

So it is not defined at $x = 0$

So note that $\cos a \frac{\pi}{2}$ is not defined

So we cannot find $g(0)$ but we can try to find the limit but limit of $g(x)$ as x

approaches 0 is equal to 3 minus limit x approaching 0 of $f(x) \times \cos x$ $x \rightarrow 0$ I can write as $f(x) \times \sin x$
 So now we have this is 0 by 0 form now we know that the limit of $\sin x$ by x as x approaches 0 is 1
 So this I can write as three minus limit x tends to zero of $f(x) \times x$ divided by $\sin x$ by x now the denominator we know the limit is one
 So this is equal to three minus limit x approaching zero of $f(x) \times x$ this is because limit of $\sin x$ by x as x approaches zero is equal to one now somehow we have to make use of the derivative of f at 0 is given to us which is 1.
 So this is equal to 3 minus limit x tends to 0 of $f(x) - f(0)$ by $x - 0$ this is because $f(0)$ is given to be equal to 0.
 So now we know that this limit is nothing but the derivative of f at zero
 So this is three minus $f'(0)$ and $f'(0)$ is given to be equal to one
 So this is 3 minus 1 which is equal to 2.
 So the answer is limit of x tends to 0 of $g(x)$ is equal to two
 So I would like to make one comment here
 So note that here we have this limit of $f(x) \times \sin x$ which is of zero by zero form
 So you might think of applying L'Hopital rule directly here
 So if we apply L'Hopital rule then this will be equal to limit of $f'(x) \times \cos x$ and then limit of $\cos x$ as x approaches 0 is 1
 So that is limit of $f'(x)$ as x approaches 0 and then you can think of writing that as $f'(0)$
 So you will directly get this 3 minus $f'(0)$ but note that to write limit of $f'(x)$ as x approaches 0 equal to $f'(0)$ we need to know that f' is continuous at 0 but if you see the problem it is only given that f is a differentiable function but the derivative need not be continuous
 So therefore that is not the correct reasoning
 So so therefore we have written this limit as this derivative at 0 and then evaluated this
 So this finishes problem number four ah let's move to problem number five which is slightly different from what we have done
 So far
 So here we are given if i is equal to summation of k equal to 1 to 98 of integral of $k + 1$ by $x \times x + 1 dx$ from k to $k + 1$ then we have to choose correct options from the following 4 options
 So a is i is greater than natural log of 99 b is i is less than log 99 c is i is less than 49 by 50 and d is i is greater than 49 by 50.
 So of course here if you see this integral of $k + 1$ by $x \times x + 1 dx$ this can easily be evaluated because this is say one by $x \times x + 1$ can be written as one by $x - 1$ by $x + 1$ and then you can evaluate this integral and actually you can get an expression for i but using that getting these inequalities might be difficult
 So what in this kind of problems we should try to look at the options and then try to see how to do it
 So if you see the option a and b then i is compared with natural log of 99
 So now note that what is log 99 note that log of 99 is nothing but integral of $1/x dx$ from 1 to 99 because the anti-derivative of $1/x$ is $\log x$
 So we get this and this can be written as summation of k equal to 1 to 98 of integral k to $k + 1$ of $1/x dx$ because for k equal to 1 you will get integral from 1 to two for k equal to two it is integral from two to three and
 So on up to integral from ninety eight to ninety nine
 So this sum will be equal to integral from one to ninety nine
 So now if you see we have written this log 99 in terms of summation for k equal to 1 to 98 of some integral
 So we we have to just compare the given integral with this integral to see which one is bigger
 So now note that if $m > x$ and $x < k + 1$ then here we had integral of $k + 1$ by $x \times x + 1$ then we have the $k + 1$ is less than $x + 1$ which implies $k + 1$ by $x + 1$ this will be less than 1
 So therefore $k + 1$ by $x \times x + 1$ this is less than $1/x$
 So integral of k to $k + 1$ of $k + 1$ by $x \times x + 1 dx$ this is less than integral of one by $x dx$ from k to $k + 1$

So this shows that summation of k equal to one to ninety eight integral k to k plus one this is less than $\log 99$ this was equal to i

So therefore a is false and b is true

So this i is less than $\log 99$ is true this one is false now c and d compares i with 49 by 50.

So in fact if you see the way we have done this part to get that the integral is less than $\log 99$ we can also get a lower bound

So i greater than something by using

So similarly what we can get is this k plus one by x is greater than one if x is between k and k plus one

So here we are using k plus one by x is greater than one

So therefore k plus one by x times x plus one this is bigger than one by x plus one and now if i integrate this an integral of k to k plus one k plus one by x times x plus one dx this will be greater than integral of k to k plus one of 1 by x plus 1 dx and therefore i which is summation of this k equal to 1 to 98 of integral k to k plus one this is bigger than integral from one to ninety nine of one by x plus one dx

So just like we got that i is less than integral of one by x dx from one to ninety nine here we are getting its bigger than one integral of one by x plus one from one to ninety nine and this is equal to $\log x$ plus one from one to ninety nine which is equal to \log of hundred minus \log two which is equal to \log fifty

So now if you see we have to compare with forty nine by fifty now clearly \log fifty is bigger than one which is bigger than forty nine by fifty

So i is bigger than 49 by 50 this is correct

So therefore d is correct and c is false

So we get d is right and this c is on

So in fact if you see we have proved that i is bigger than natural log of fifty which is much bigger than this number given 49 by 50.

So you you might wonder how to get this number forty nine by fifty

So if you see we have summation from one to ninety eight

So this forty nine by fifty is nothing but ninety eight by hundred

So so let me write here remark note that this forty nine by fifty is equal to ninety eight by hundred

So if we can

So that this each integral is bigger than one by hundred then the summation will be bigger than ninety eight by hundred

So let us look at the integral if we saw that integral k to k plus one this is bigger than one by hundred for each k then i will be bigger than ninety eight by hundred which is forty nine by fifty showing that this integral is bigger than one by hundred is not difficult

So what i will do is we put x is equal to k plus y

So what happens if i put x equal to k plus y then when x is between k and k plus one when x is equal to k y will be equal to zero and when x is equal to k plus one y is equal to one

So we can write this integral as integral this is equal to integral from zero to one now and k plus one x is equal to k plus y times x plus one is k plus one plus y dy now because y is varying between 0 and 1 then k plus 1 is bigger than k plus y

So this is bigger than integral from 0 to 1 1 by k plus one plus y dy this is since k plus one by k plus y this is bigger than one and now if you see this integrand one by k plus one plus y this is bigger than one by k plus two and the integral is from zero to one dy that gives one

So this integral is bigger than one by k plus two now k varies from one to ninety eight

So the this is always bigger than equal to one by hundred if k is between one and ninety eight

So we have proved that this integral is bigger than one by hundred in fact we can get a better bound by using this is bigger than one by k plus two and then summing it from k equal to one to ninety eight

So we can get that this d is true by this way also ok let us do one more problem

So question number six

So we are given a function let f r to r be defined by f x equal to greatest integer of x if x is less than equal to two and zero if x is greater than two now if i is equal to integral from minus one to two x times f of x square divided by two plus f x plus one dx

then we have to find the value of i

So to do this problem first let us recall that the greatest integer function this is the greatest integer less than or equal to x

So this is equal to n if x is greater than equal to n and strictly less than n plus one for any integer n

So we have if you see f of x is defined to be greatest integer of x if x is less than equal to 2 and 0 for x bigger than 2 now i is integral of this

So let us write g of x is equal to x times f of x square divided by two plus f of x plus one now f of x square is greatest integer of x square if x square is less than equal to two and zero if x square is bigger than two what is this equal to

So x square is less than equal to 2 means x between minus root 2 to root 2 it is given by greatest integer of x square now greatest integer of x square this will be equal to 0 if x is between minus one to one because in this case x square will be strictly less than one

So greatest integer will be zero and if x is greater than equal to one and strictly less than root two then x square is bit greater than equal to one and strictly less than two

So the greatest integer of x square will be one and if of course x is bigger than root two then x square is bigger than two

So this is equal to zero

So because we have to integrate from minus one to two

So that is why i have started only for x bigger than minus one and f of x plus one if you see this is equal to greatest integer of x plus one if x plus one is less than equal to two and zero if x plus one is bigger than two

So this is equal to greatest integer of x plus one if x is between minus 1 x is less than equal to 1 and this is 0 if x is bigger than 1. now again this greatest integer of x plus 1 this will be equal to 0 if x is less than zero and greater than equal to minus one and if x is greater than equal to 0 and less than 1 then x plus 1 is greater than equal to 1 and less than 2

So the greatest integer will be 1 and this is equal to 0 if x is bigger than one

So now we can write g of x this was x times f of x square by two plus f of x plus one

So if you see f of x square is non zero only between one and root two

So if x is greater than equal to 1 and less than root 2 then this is x times f of x square is 1. divided by two plus f of x plus one is because x is bigger than 1 this will be equal to 0 and this is 0 if x is bigger than root 2.

So g of x is equal to x by two if one is less than equal to x less than root two and zero otherwise

So now i is equal to integral of minus one to two of g of x dx this we write as integral from minus one to one g of x dx plus integral of one to root two g of x dx plus integral root 2 to 2 g of x dx now here g of x is 0 in the first interval and the last one

So this is equal to 1 to root 2 of x by 2 dx which comes out to be x square by four evaluated between one and root two and this gives answer is one by four times two minus one which is one by four

So i is equal to one by four is the answer all right

So this finishes lecture one on integral calculus in the next lecture we will discuss some more problems on integration thank you you