

welcome students in previous lecture you have seen about the properties of indefinite integral as well as method to evaluate sum of indefinite integrals ah most of the integrals initially we evaluated using the idea of anti derivative or from the differential calculus derivative ideas and the later half we found that it may not be always possible to get integrals ah using those ideas and therefore we have to have some more tools

so for that we used another method known as method of substitution which we saw that it it will convert our integrals into simpler forms which can be evaluated easily towards the end i took an example for you integral of sine of $a x + b$ $d x$ and by substitution of $a x + b$ equals to t and then finally evaluating it we saw that the answer comes to be cosine of $a x + b$ by a plus another constant c is the constant of integration this we saw by making the substitution $a x + b$ equals to t i may recommend that this formulation can be actually generalized into let us say if we know the integration of $f x$ to be say capital x ignoring the constant then we can write the integration of $a x + b d x$ as f of $a x + b$ divided by a

so now if we choose constant we know that the constants will take care of each other i will show you with an example also

so in this case if i am given a function of this kind then i can always write ah integral for this kind

so that simply if i impose the formulation here i know that the integration of sine x is minus of cosine x and therefore using this formulation the integration of sine of $a x + b$ becomes minus of cosine $a x + b$ by a plus constant

so the proof of this is not very difficult one can see that why this is happening is that that in this star equation choose $a x + b$ equals to a new variable t

so that you know $a d x$ turns out to be $d t$ or $d x$ becomes one by $a d t$ and therefore the integral will be represented as integral of $f a x + b$ is t therefore it is f of $t d x$ is nothing but $d t$ by a

so it will be equals to 1 by a integral f of $t d t$ now t can be replaced with any variable and therefore the form is same as $f x d x$

so you can even write it in this manner it is not necessary to write but you can write it even in this manner that 1 by $a f x d x$ and this form has the the integral as capital f

so that will make 1 by a capital f but you have to mind that this is corresponding to this f function

so corresponding f function has to be again converted to the original function

so it is better if we if we keep the original variable t itself

so that it should not confuse you

so this will give me f of t plus a constant of integration and since already this t variable is is there with a value of x $a x + b$ i can write it f of $a x + b$ by a plus constant of integration which is same as the the right hand side which is given here

so ultimately what we we learnt is that that if we know the integral here of function with respect to the variable then if you have a linear function here then what you can get is that that substitute the integral value for that function and then divide by the differential of the term which is here which is equals to a this is many times useful ah in evaluating integral we do not have to do calculation again and again though initially we will do those calculations now i will show you one example let us say integral of sine of $a x + b$ into cosine of $a x + b d x$

so i can evaluate this integral by using substitution

so let us see how it evolves and another way is that as i as i shown you in the previous example that what if i already know the integral of $\sin x \cos x d x$

so this is my $f(x)$ this is $f(ax + b)$ this evaluation i can make by dividing and multiplying it by two

so that will become $2 \sin x \cos x dx$ and then this i can write as half of $\sin 2x$ integral of $\sin 2x dx$ and again the same idea of $\sin(ax + b)$

so i know the integral of $\sin x$ that is $-\cos x$

so $-\cos$ instead of x i have to put $2x$ and then divided by the differential here

so $2 + \text{constant}$

so now i know integral of $\sin x \cos x dx$ which is $\frac{1}{4} \cos 2x + \text{constant}$ and if i want to relate the integral with this i using my previous formula in this case i should come out to be $-\frac{1}{4} \cos$ of 2 mind it the variable here is x and it is coming out to be $2x$ therefore 2 and then $ax + b$ divided by the differential of this term which is $a + \text{constant}$

so this gives us the result for the integral now what if we have chosen we have made a choice of substitution $ax + b = t$ then the integral i will turn out to be integration $\sin t \cos t dt$ by a because from here you can find that $a dx = dt$

so this number will be $\frac{1}{a} \int \sin t \cos t dt$ and then dt integral $\sin t \cos t dt$ again using this same procedure or even again using another substitution of choosing $\sin t$ to be let us say another variable u we can make another substitution $\sin t = u$ which will imply that $\cos t dt = du$

so that $\cos t dt = du$

so this integral will become $\frac{1}{a} \int \sin t \cos t dt = \frac{1}{a} \int u du$

so you will get $\frac{u^2}{2}$

so this will be $\frac{u^2}{2a} + \text{constant}$ what is u $u = \sin t$ what is t $t = ax + b$

so substituting back all the values we will get $\sin t$ that is $\sin^2 t$ and place of t i should substitute $ax + b$ divided by $2a$ and plus a constant

so what did we get look at these two values are the same

so proceeding with two different formulations are two different formula two different methods what we realize is that the functions which we get as integrals they look different but actually they are not the similar assumption we have shown in previous lecture that integral of a function is not unique it is unique only up to a constant

so that the problem here is that the function $-\frac{1}{4} \cos^2(ax + b)$ plus c belongs to the same set of family of the function which is $\frac{\sin^2(ax + b)}{2a} + c$ i'll show you how it can be done what you do is that that you can realize if you use the trigonometrical formula for $\cos^2 \theta$ you know that is equals to $1 - 2 \sin^2 \theta$

so therefore if i substitute $\cos^2(ax + b) = 1 - 2 \sin^2(ax + b)$ i will see that that 2 will cancel with this 2 here which is 4 here

so this 2 will make it 2 by 4 will make it 2 here and therefore this negative sign will make it positive

so what we should get is $\frac{\sin^2(ax + b)}{2a}$

so the the variable term i will get as same as here and the constant term which is appearing here will be $-\frac{1}{4}$ which i did with this constant c will be another constant c_1

so if we if we if we look at the two results though they look a different form but they are same

so many times it may happen that you may choose a different path for evaluating the integral and you may get a form which may not be exactly the same as you

have gotten ah from from a different formulation but ah the the results the family of functions they will represent that will be same now ah next example which i will show for you is let us say we take integral of a

so we have to evaluate this integral now one can make a choice immediately for choosing the function which we should substitute because we cannot it is evident very clearly from here that it is not ah having anti-derivative which we can immediately recognize and therefore we have to substitute first of all

so one can make a choice first as root x to be a substitution as t that means x raise to power half is equals to t

so one half x raise to power minus half d x is equals to d t which is one by two root x d x is equals to d t

so the integral will take form as tan raise to power four t sec square t and then dx by root x will become 2 dt

so this root x will be actually used here

so d x by root x it is 2 d t now further it it cannot be immediately solved unless until we make certain more substitution

so we see that that sec square is appearing here and tan is appearing here we know that the derivative of tan is sec square and therefore it should click us immediately that we must choose another substitution here as tan t

so while making a choice of substitution we should look at at the function ah which is which is in the integrand that any part of integrand can be represented as the derivative of some part of the integrand if it is possible then that part should be chosen for example here tan t if we differentiate tan t you will get sec square t

so sec square t is part of the integrand and therefore if we choose tan t then sec square t dt can be represented as another new ah differential of new variable

so we are making a choice of tan t is equal to let us say u

so that seck square t d t is equals to d u making this substitution we note that this becomes u raised to power four sec square t d t two is already outside is d u and we get twice of u raised to power five by five plus constant which after substitution of the value u is equals to tan t t is equals to root x we get two by five tangent power five of root x plus constant alternatively one could have also seen here directly the same logic which i gave you here is that that tan function is here and sec square is already appearing here

so it's possible immediately to assume this tan function as as a new variable that means can we take tan of root x as new variable t

so let us see what will happen if we choose it

so tan of root x if we differentiate we will get sec square root x and the differentiation of root x will give you one by two root x d x which is equals to d t simplification of this will give you sec square x sec square root x d x over root x two goes to the right hand side equals to two dt why i am writing it in this way because this is the part of the integrand sec square root x over root x is the part of the integrand here

so if you notice it here

so sec square root x over root x is part of the integrand which if i make a choice of tan root x equals to t it will give me a simplified function

so ultimately what i will get as integral i equals to tan root x is already chosen to be t and therefore this will become t raise to power 4 into 2 dt

so 2 will turn out to be outside this will become 2 by 5 t raise to power 5 plus constant which is same as two by five tangent of five root x plus c

so you will get the same form of the of the solution if you make a choice but if you notice in this case the computation was far more lesser than in this case because we have to make choice two times once here we have to substitute and

once another time we here we have to make a substitution

so when we are may using this method of substitution then it is important to make correct choice ah a better choice may lead to very ah quick calculation and will lead you to the correct answer quickly now will quickly do couple of little complicated looking examples say for example one i choose the function to find out the integral of $x^4 \sin(\tan^{-1} x)$ raised to power 4 divided by $x^8 + 1$ dx

so this is the integral which we need to find out if you look at this integral this looks a very complicated form x^4 then ah sine then tan inverse these functions are appearing there but if you look at the the the terms carefully and the logic which we understood in the previous problem is that is there a part of integrand which we choose as a as a new variable then is it possible to have the derivative of that part within the integrand itself it may be very clear from here if you look at first of all x^4 is there the derivative of x^4 is x^3

so one substitution can be x^4 which will give you x^3 as the derivative and the next term you further go for substitution another term if you look at is the tan inverse present and you know the derivative of tan inverse takes you to the square of the variable plus one

so if i look carefully the term tan inverse x^4 raised to power four the term here is one ah one plus x^8 which is the square of this term therefore it looks very rational that if i make a choice of tan inverse x^4 raised to power four equals to new variable t then this will give me four x^3 four x^3 into one by $1 + x^8$ dx is equals to dt which after little bit of simplification i can write that $x^3 dx$ is equals to dt by 4 which is here the part of my integrand

so so i will make the replacement of this

so that i get i equals to integration instead of dx x^3 raised to $x^8 + 1$ i will write here ah dt by 4 then this is sine of new variable t sine of t you can see this simply it is ah minus of $\cos t$ plus the constant which ultimately will take you to minus of one by four \cos of tan inverse x^4 plus constant

so problem is solved quickly by making a right choice of the function and ah we immediately get the result another example for you is this one

so let us say that we want to evaluate the integral i equals to 10^{5x} raised to power x multiplied by 5^x multiplied i mean of the integral of dx

so here the integrand is ten raised to the power five raised to the power x into five raised to power x what should we do

so if you look at this problem this can be immediately understood that if we use 5^x as a substitution for new variable t and use the formula of a raised to the power x square is some constant

so the derivative or the differential becomes a raised to power x log a base e dx equals to dt

so that if i make the choice of five raised to power x as new variable then 5^x raised to the power x is already presented as a part of integrand which along with dx will make as dt

so this integral can be written in this form 10^{5x} raised to the power x will become new variable t five raised to the power x dx five raised to the power x dx will be dt divided by $\log 5$ b c and then ultimately this integrand can be written very easily 1 by $\log 5$ base e integral of 10^{5t} raised to power t which you can write as 10^{5t} raised to power t integral of a raise to power x is integral is a raise to power x divided by \log a base e

so it will be giving me divided by $\log 10$ base e plus constant of integration i can substitute this value for t here ah using this this t substitute for five

raised to power x

so that i can get the the real answer

so with the help of these examples we have

so far learnt that ah how to make choices what kind of choices can we make you can make few choices and in one case it may happen that it may ah i mean the problem may be a little longer but ultimately it will give you the solution

so don't be disheartened at the beginning after some practice you will understand that which function you should choose

so that you can easily compute the integrals now we shall look for integrals of some important functions which are involving some trigonometric functions

so these functions once evaluated we will use them as formula

so for example

so far if we need to evaluate integral of $\tan x$ we do not know what to do but now we have a tool which will help us to understand how to evaluate integral of $\tan x$

so we can see how it can be done first of all we know that $\tan x$ is $\sin x$ by $\cos x$ and now look at this function look at the integrand $\sin x$ and $\cos x$

so if i make a choice of one of the function as a new variable then i can see the derivative of that function is present here but which which function i should choose

so that the product with that of that function with derivative becomes a new variable

so if you look carefully you can see that that if i choose $\cos x$ as a new variable i will get as $\sin x dx$ as a ah dt with a negative sign of course

so my choice is very clear from here by closely looking at this term

so i will make this choice $\cos x$ to be new variable t

so that minus sign $x dx$ becomes dt and the $\sin x$ is part of the integrand therefore ah the integral let me call this integral as i

so the integral i becomes integration of minus dt over t this minus sign goes here and then $\cos t$ $\cos x$ is t

so it comes this way dt over t is i already know this is a log function this we already know from the formula

so \log of $\text{mod } t$ and plus constant negative we know minus of \log

so it can be converted into \log of one by ah $\text{mod } t$ and therefore ah this formulation will take you to first minus of $\log t$ is equals to $\cos x$ plus c which ultimately because of this negative sign will lead you to \log of mod of $\sec x$ 1 by $\cos x$ is $6 x$

so what we got here is that integral of $\tan x dx$ is equals to \log of $\text{mod } \sec x$ plus constant

so this we shall use as formula in the similar fashion we can use the integral of $\cot x$ by doing the same process $\cos x$ divided by $\sin x$ and writing all the calculations here you can find out that this will be \log of $\text{mod } \sin x$ plus constant

so this part you can prove yourself ah you do it yourself it can be very easily done by writing on the same line as previous formula third formula is for integral of $\sec x dx$

so we will call it as i now $\sec x$ is you know one by $\cos x$

so there is again problem there is no part apart from the function say $k x r$ one by $\cos x$ two in that case $\cos x$ itself

so what should we do

so the idea is simple what we do here is that we try to convert it into a form where we can ah where we can use our or where we can utilize our previous results

so for that what we do is that we multiply both numerator and denominator by

sec x plus tan x it will be clear in a moment that what benefit you will get if you do that

so now if you expand and write the function

so it will become sec square x plus sec x tan x whole divided by sec x plus tan x now look at this function carefully what benefit did you get

so if you look at the denominator it is sec x plus tan x and if you differentiate that function the differentiation of sec x will give you sec x tan x and differentiation of tan x will give you sec square x and look at now the numerator both of them are the same function right

so sec x plus tan x if you differentiate you are going to get the numerator which is the part of the integrand and therefore our previous trick if i assume sec x plus tan x to be new variable t we see that sec square x plus sec x tan x dx equals to dt and therefore this term can be written as dt over t which makes life very simple now leading it to logarithmic of mod t plus constant and what is t this is log of sec x plus tan x plus constant

so the integral of sec x becomes this formula on the parallel lines we can actually quickly evaluate the integral of cosec x i hope now you understood the trick is that that we have to multiply and divide by cosec x plus cot x and look at the terms again carefully

so if you choose cos x x plus cot x as a new variable you will get minus of cos x square x and then minus of cos x x cot x that will make both of them as a sum and therefore ah the part of the derivative is present in the integrand

so we will make the choice say choose cos x x plus cot x as new variable again let us say t

so that cos x x cot x minus minus cos x squared x whole multiplied by dx is equals to dt

so if i take this negative sign as common i am getting the same thing as it is here cos x square x plus cos x x cot x and that will be simply written as minus of integral dt over t ah which is nothing but minus of logarithmic of mod t plus constant ah t is again causing x plus cot x

so it will become i can write it this way log of mod of one by cos x x plus cot x plus constant c ah normally the formula is written further by simplifying not keeping the term in the denominator and it is written ah in terms of another function which which can be easily seen from here what we do is that we multiply both numerator and denominator by cos x x minus cot x

so that the numerator becomes ah cos x square x minus cot square x ah or you can even think that this one i can replace with cos x square x minus cot square x and then factorize it

so after those calculations ah which however you think it off you will get it as log of cosec x minus cot x plus c

so the integral of cos x x dx equals to

so so this equals to here

so hopefully you understood the point here this is one over cos x plus cot x one way is that that you multiply both numerator and denominator here by cos x minus cot x

so numerator you will get cos x x minus cot x and denominator you will get cos x square x minus cot square x and you know that cos x square x minus cot square x equals to one

so ah we have seen that we have developed some more ah formulas for the integral of some trigonometric functions ah these formulas will again be handy ah later on when we evaluate the integrals i will show you with help of one example integral of sine x over sine of x plus a dx

so if you look at this example ah again we have difficulty we cannot see a term which if we choose as a new substitution then we get ah some term from the

integrand but we can make a choice here ah if we look at it carefully what is happening is that this sum is coming in the denominator if somehow we shift it to the numerator it may help us

so what if we replace this $x + a$ in the denominator by a new variable t so this will give us $dx = dt$ and therefore the integral will be converted into this form $\sin(x)$ will become $t - a$ and $\sin(x + a)$ this will become $\sin t$ and dx is dt

so the integral this is converted into this form now this may help us why because we know the formula of $\sin(a - b)$ which we can expand and we can see what can happen

so we do

so this becomes $\sin t \cos a - \cos t \sin a$ divided by $\sin t dt$

so this calculation leads to $\sin t$ gets cancelled in the first integral and if you use the linearity property $\cos a$ being a constant comes out of the integral it becomes $\int dt - \sin a \int \frac{\cos t}{\sin t} dt$

so we replace this as $\cos a$ integral of one becomes t plus let us say another constant c one minus $\sin a$ this is $\cot t$

so just now we have evaluated the integration of $\cot t$ integration of $\cot x$ we evaluated

so this is as i told you the same material what variables is being used we can simply modify it

so this will become \log of \sin

so instead of x here you should get t plus constant

so this is the formula which we are going to use

so this will be \log of $\sin t$ and plus another constant let us call it as c two

so ultimately ah the integral is done and now just we need to put some simplification t is say $x + a$

so we will put it as $x + a + c$ one into $\cos a - \sin a$ this \log of $\sin(x + a + c)$ which further can be adjusted with the constant like $a + c$ one of $\cos a$ then $\sin a$ times c two this whole term can be written as a new constant c

so that we ultimately get $x + \cos a - \sin a$ times \log of $\sin(x + a + c)$ plus the whole constant c

so sometimes we have to use a kind of what you can call is a trick

so that the integral which initially may not be of the form which we can easily evaluate but if we make a substitution it may be converted into another form which we can which can easily evaluate and knowing this formula became handy here and we could be able to evaluate this integral quickly in the next we are going to use certain trigonometric identities

so first example which i would like to put in front of you is $\int \sin^3 x \cos x dx$ i have just chosen three

so that ah it will not be very complicated calculation and we can easily do this one if you look at it carefully again we can use the method of substitution we have the sine function here and the cosine function here

so what we can do is that we can break it into $\sin^2 x \cos x dx$ that's how we should look at it why we are doing this because we know that the derivative of $\sin x$ is $\cos x dx$

so at least one part is gone and now we have to take care of the rest of the part and what we should do with this $\cos^2 x$

so we should try to can we can we relate this function in in sinus sine term and here you know you have a trigonometrical identity which says that $\sin^2 x + \cos^2 x = 1$ that means that $\cos^2 x = 1 - \sin^2 x$

$\sin^2 x$

so you can convert this also into a sine function and that is how this whole thing goes

so $\sin^3 x$ into $\cos x dx$ now if i make a choice of $\sin x$ to be t choose $\sin x$ to be t

so that $\cos x dx$ is dt what i get here is that $t^3 - t^2 + t - 1$ simple polynomial expression which can be easily evaluated $t^3 - t^2 + t - 1$ raise to power five which will make it $t^5 - t^4 + t^3 - t^2$ raise to power six by six and plus the constant of integration where t is $\sin x$

so ultimately you will get $\frac{1}{6} \sin^6 x - \frac{1}{5} \sin^5 x + \frac{1}{4} \sin^4 x - \frac{1}{3} \sin^3 x + C$ constant ah its not necessary to go or to deal with this example the in the fashion in which i dealt it you can also use

so let me take the same example again you can also use some other relationship so if i write this as a product of $\sin x \cos^2 x dx$ what we can do here is that is we multiply it by two that means since we are multiplying inside the cube bracket that means we are multiplying by two cube

so we have to divide by two cube once i do this this looks to a very ah familiar formula $2 \sin^2 x \cos^2 x$ is that $\sin^2 x \cos^2 x$

so this makes integral one by eight can be taken outside $\int \sin^2 x \cos^2 x dx$ and here i can make use of my substitution for $2x$ in order to evaluate the $\sin^2 x$ function

so let us say that $2x$ is t so that twice of dx is equals to dt

so that dx is equals to dt by two

so after making the substitution i will get here one by eight integration of $\sin^2 t dt$ by two which will make it as one by sixteen integration of $\sin^2 t$ i am doing all the steps

so that it should be very clear to you well $\sin^2 t$ ah as such we do not know any formula at this moment what should we do should we convert it into another \sin^2 into $\sin t$ and then go with the the process which we have already done or should we use another trigonometric identity hopefully you remember the formula $\sin^2 x = \frac{1 - \cos 2x}{2}$

so if i use this formula from here i can easily get that depends if i remember the formula then only i can use this one

so if i remember the formula i can immediately get $\sin^2 t$ as $\frac{1 - \cos 2t}{2}$ variable is t $\int \sin^2 t dt = \int \frac{1 - \cos 2t}{2} dt = \frac{1}{2} \int (1 - \cos 2t) dt = \frac{1}{2} (t - \frac{\sin 2t}{2}) + C$ which will lead to one by sixteen integral of $\sin^2 t$ is $\frac{1}{16} (t - \frac{\sin 2t}{2}) + C$ again minus $\cos 2t$ by two and then plus a constant C

so ultimately what i have observed here are what i have found here is the following minus three by sixteen $\cos 2t$ is $2x$ therefore $2x$ this minus becomes plus one by sixteen into three $\cos 2x$ plus constant

so this is what we are going to get well the forms which you look at again in the previous example this example the way we evaluated they may not be looking to what you were expecting but as i told you earlier also that the uniqueness is not ah guaranteed but if you see they belong to the same family you can expand this $\cos 2x$ into ah one minus two $\sin^2 x$ and then again further

so that you will get the same terms

so ah towards end i would like to show you another example which uses some trigonometric relationship to find out the integral of $\sin^4 x$ sign eight x in fact these examples they can be considered as ah generic examples

so in case ah you have to solve or you have to evaluate some other integral which will involve these kind of functions you should take message that how you can deal with those functions

so for example here again as in previous example you multiply and divide by two

so that you will get one half of two sine four x sine eight x if you look at this integral now this has taken the form two sine a sine b fortunately we have the formula to sign a sign b you remember the formula this takes you to sine of sorry takes you to cos of cos of a minus b minus cos of a plus b

so here our integral i in this case will become one half integral to sign a sine b cos of four minus eight x minus cos of four plus eight x

so once i use the formula you can see that this entire evaluation becomes trivial one half integral cos of minus four x which is cos of minus x is always cos x

so we shall have cos of four x minus cos of twelve 12x dx and this using the relationship which i already showed you for a x and v that if i know the integral of cos then i just need to ah divide that by ah by the derivative of this term that means that cos has the integral sign

so it will become four x divided by the derivative of this fellow four minus cos twelve has integral sign twelve x divided by the derivative of twelve x is twelve and then plus a constant of integration

so this is what we get

so looking at this particular expression if you look at it it looks a bit difficult but ah using the trigonometric formula

so trigonometrical relationships helps us to evaluate ah the integrals in a better way or ah in a simpler way

so towards end i would like to summarize what we have done

so today we learnt how to evaluate indefinite integrals by substitution then for some trigonometrical relationship or trigonometric formula and further we use trigonometric identities to evaluate these integrals in next class we will see how to evaluate certain other specific functions which will involve algebraic expressions or the polynomial expressions you