

welcome students in previous class we understood the idea of indefinite integrals we understood what it meant by an area function we also understood what is the idea of anti-derivative then we defined the integral of  $f(x)$  as collection of all anti derivatives this  $c$  belonging to a set of reals towards end we understood the geometrical interpretation of this family of curves or the anti derivative and we also looked at the area function from the perspective of integral considering it as integral which we said that this is a definite integral

so these are the things which we learnt in our previous class today we will look at further that what are the properties of indefinite integrals how can we evaluate the integral of a function which will be given to us

so before we start we take one example where i will show you the importance for the constant  $c$  which we take when we take the integral

so suppose that that we are asked to find out anti-derivative  $f(x)$  of function  $f(x)$  which is given as  $5x^4 + 2$  such that the value of anti-derivative is given to us

so it not only says that we have to find out the anti derivative of  $x$  but also it says that that we have to find out that nt derivative which has the value  $f$  value as five when  $x$  equals to one

so out of the family of all the anti derivatives you have to look for a particular anti-derivative i will show you how it can be done

so from our previous experience of anti derivative of a function of the type  $x$  raise to power  $n$  we can figure out that  $d/dx$  of  $x$  raise to power five plus two  $x$  turns out to be and therefore this function is considered as the anti-derivative we need the anti-derivative such that  $f(1) = 5$  which will mean that one plus two plus  $c$  equals to five giving the value of  $c$  as two

so the anti-derivative  $f(x)$  from here gives the value  $x$  raise to power five plus two  $x$  plus two

so in this case you notice that that we have got a unique anti-derivative  $x$  raised to power five plus two  $x$  plus two which gives you the value when  $x$  is equals to one as five

so if we are given a condition such that the condition which was given in this case as  $f(1) = 5$  the specific value of that constant can be evaluated in general we consider this constant as any arbitrary constant next we shall look at the properties of indefinite integrals the first property relates as we told initially that integration can be considered as inverse process of differentiation

so the property  $d/dx$  of  $\int f(x) dx$  is the function itself that means that it the the integral of the function and if you take the derivative

of that integral you will get the same function second property is that  
integral  
of the derivative of the function is the function plus a constant now look at  
these two expression first  
before we go for the proof of these equations the first expression says that  
the differentiation  
of a integral of a function is the function itself while in the second case it  
says  
the integral of the differential of the function is function plus constant  
and therefore we do not say that the two operations the differential  
and  
the integrals are inverse operation to each other but we say that they are  
they  
can be thought of as an inverse operation because had the been inverse  
operation then after  
application of both the operations simultaneously they should have given you  
the function itself  
but here in this case it is a constant  
so if we consider uniqueness up to the constant then  
they can be thought of as a inverse operation proof  
so property a can be  
directly proved using definition we know from the idea of  
anti-derivative is that  $d/dx$  of  $f(x)$  equals to small  $f(x)$  such that capital  
 $f(x)$  is anti derivative of small  
 $f(x)$  then  $\int f(x) dx$  is capital  $f(x) + c$   
so now we will apply the derivative operator  
on this integral assuming that capital  $f$  is the anti-derivative for small  $f(x)$   
so if we  
apply the derivative here it will give us  $d/dx$  of  $\int f(x) dx$  will be  
same as it  
applied on the right hand side  $d/dx$  of  $f(x) + c$  which is same as  $d/dx$   
over  $d(x)$  because the constant  
derivative of the constant will be zero  
so and we already know from this relationship  
that  $d/dx$  is nothing but  $f'(x)$   
so  $d/dx$  of  $\int f(x) dx$  turns out to be  
the function  $f(x)$  itself  
so  $d/dx$  of  $\int f(x) dx$  is function itself  
so this shows property  
a for property b we again use the definition  
so we note that  $d/dx$  of  $f'(x)$   
 $x$  which is basically  $f''(x)$   
so now if you look at it again from the  
perspective of the definition of nt derivative then small  $f(x)$  is anti derivative  
for  
 $f'(x)$  and therefore using the definition of integral we can write that  
integral of  $f'(x) dx$  is  $f(x) + c$  for  $c$  belonging to the set of  
reals and this is what the same as property b  $f'(x) dx$  is equals to  
 $f(x) + c$   
so we have shown this property using definition that integral of a  
derivative is same function plus a constant second property which we look at is  
that if we  
are given that the derivative of two integrals are same that means that for two  
functions  $f(x)$  and  $g(x)$  if the  $d/dx$  of  $\int f(x) dx$

$x$  is same as  $d$  by  $d x$  of integral  $g(x) dx$  then both the functions integral  $f(x)$  and  $g(x)$  they belong to same family of functions how can we if we show that so what we do is that we take the entire expression the expression implies that  $d$  by  $d x$  of integral of  $f(x) dx$  minus integral of  $g(x) dx$  zero so first you transfer this in  $d$  by  $d x$  of integral  $d x dx$  to the left hand side then take the operator outside and then write this expression in this fashion since this equality is true for all  $x$  and therefore this equality is also true for all  $x$  and this is possible that the derivative of some function of  $x$  equaling to zero the possibility is only if the function itself is a constant which means that integral of  $f(x) dx$  minus integral of  $g(x) dx$  is equals to a constant let us say  $c$  and which will mean that that if i transfer  $g(x)$  on the right hand side then the collection of all functions integral  $g(x) dx$  plus  $c$  one let us call this as  $c$  one such that  $c$  one is in  $\mathbb{R}$  and similarly if i take the function  $f(x)$  on the right hand side then integral  $f(x) dx$  plus  $c$  two such that  $c$  two belonging to  $\mathbb{R}$  so they represent the family of curves for for these integrals and therefore the two integrals because these two families because of this equality here they are equivalent so we usually do not as as these families are equivalent we usually do not bother about the constants which are written here and we write that integral of  $f(x) dx$  is same as integral of  $g(x) dx$  that means the constant  $ah$  is is omitted here further we look for some more properties of  $ah$  integral operator these properties are similar to the property of differential operator which you have already seen the first property is linearity property which i will write in this in the in the following manner integral of  $f(x) + g(x) dx$  is same as integral of  $f(x) dx$  plus integral of  $g(x) dx$  look at the functions it says that the integral of sum of two functions is same as sum of the integrals of those two functions so integral basically distributed on the on the sum proof is simple what we do here is that that you take left hand side and differentiate it  $d$  by  $d x$  of the left hand side gives you as we know from property one which we have already shown is that the derivative of integral is the function itself so using the property even which we have already proved is that the derivative of this integral is nothing but  $f(x) + g(x)$  this we say that is relation one which is coming from the left hand side now similar thing we will do on the right hand side differentiate the right hand side as we already know that derivative is

distributive

on addition and therefore we can write it as  $\frac{d}{dx}$  on integral of  $f(x) dx$  plus  $\frac{d}{dx}$  of integral on  $g(x) dx$  now again using property one we know that

$\frac{d}{dx}$  of integral is same as the function  $f(x)$  plus integral  $\frac{d}{dx}$  of integral  $g(x)$  is

the same as function  $g(x)$  consider is at relation to

so what we have shown here is that that if we differentiate the two functions on the left hand side and right hand side we get the same derivative and from the property previous property they belong to same ah family of curves

and therefore this property is true

so this proves that the linearity property

on the integrals is followed second property is for scalar multiplication

so what it says here is that that integration of  $k$  times  $f(x) dx$  is same as  $k$  times integration of  $f(x) dx$  where  $k$  is some constant this also we shall prove using the same idea

as we have done for the previous property  $\frac{d}{dx}$  of the left hand side turns out to be and the  $\frac{d}{dx}$  of right hand side we know that scalar  $k$  can be taken outside for

the different from the differential operator giving us again using the property one

so again similarly as in in previous case we we claim that  $k f(x) dx$  is same as  $k \int f(x) dx$  what we shall do now is that we shall club these

two properties together and put them in a general formula

so let us say for constants  $k_1, k_2, \dots, k_n$  and functions  $f_1(x), f_2(x), \dots, f_n(x)$  have the relationship that integration of

$k_1 f_1 + k_2 f_2 + \dots + k_n f_n dx$  is same as you take the  $k_1$  outside integrate  $f_1 dx$  plus  $k_2$  and then  $k_n f_n dx$  this property helps in evaluating the

integrals where the functions are written in certain linear combination we give one quick

example

so suppose we have to find out the integral of the function  $a x^2 + b x$

plus  $c$  to find out the integral of this function we write it in this fashion now since we

know the linearity property therefore we can write this  $a, b, c$  being constants

here  $a \int x^2 dx + b \int x dx + c \int dx$  and there is

one

so therefore we write it as  $\int (a x^2 + b x + c) dx$  we already have seen or we can even ah use the

method of anti derivative here is that that if we take the differentiation of  $x^3$  by 3 we

will get  $x^2$  and therefore this we can write as  $\frac{1}{3} x^3 + b x$  you have already seen

that this is for  $x^2$  plus small  $c$  the integration for one that means

if we differentiate  $x$  you will get one therefore the function  $x$  should appear here

and finally the constant

so we will call it as  $\frac{c}{1}$  one because this  $c$  is already appearing here so it should not confuse us

so the integral of this linear combination turns out to be this function which could easily be evaluated by breaking this integral into three separate integrals and evaluating them this technique will help us further in solving certain complicated problems now we shall use our knowledge of differentiation and write certain formulas which will help us in evaluating the integrals during problem solving these formulas are very basic formula and you should try to remember them as much as possible

so what i will do on the left hand side i will write the corresponding formula of the derivative and on the right hand side i will write the corresponding integral

so we have this  $\frac{d}{dx} x^{n+1} = (n+1)x^n$  as  $x$  raised to power  $n$  so corresponding integral of  $x$  raised to power  $n$   $x$  becomes  $\frac{x^{n+1}}{n+1} + c$  here we must note that  $n$  cannot be equals to minus 1 we will deal with this minus 1 case separately which will come in due course of time as a particular case we know that  $\frac{d}{dx} x$  is one and therefore the integral of one  $\frac{d}{dx} x$  we have already seen it turns out to be  $x + c$  certainly grammatic functions for example  $\frac{d}{dx} \sin x$  is  $\cos x$  and therefore integral of  $\cos x$  is  $\sin x + c$   $\frac{d}{dx} \cos x$  turns out to be minus of  $\sin x$  therefore we will put minus sign here so that when we write the integral it becomes integral of  $\sin x$  is minus  $\cos x$  plus the constant  $\frac{d}{dx} \tan x$  is  $\sec^2 x$  and therefore integral of  $\sec^2 x$  is  $\tan x + c$  these are all standard formulas which you can find in any of the reference book  $\frac{d}{dx} \cos x$  is again minus  $\sin x$  but i shall write it in this manner so that it becomes integral of  $\cos x$  is  $\sin x + c$   $\frac{d}{dx} \cot x$  is  $-\text{cosec}^2 x$  which gives you the integral of  $\text{cosec}^2 x$  is  $-\cot x + c$   $\frac{d}{dx} \sec x$  is  $\sec x \tan x$  and  $\frac{d}{dx} \text{cosec} x$  is  $-\text{cosec} x \cot x$  with the negative sign here which again similarly as earlier i will take it here so it becomes integral of  $\cos x \cot x$   $\frac{d}{dx}$  equals to minus  $\cos x$  plus  $c$  further we shall look at the derivative of inverse trigonometric functions so we have seen it during previous class  $\frac{d}{dx} \sin^{-1} x$  is one over square root of one minus  $x$  squared and therefore the integration for one over one minus  $x$  square square root  $\frac{d}{dx}$  is  $\sin^{-1} x + c$  constant and also we have seen that  $\frac{d}{dx} \cos^{-1} x$  is also same as square root of one minus  $x$  square and therefore one minus

so i should not get confused with the same function because we have already shown you that these two functions belong to same family of curve and therefore sine inverse x and minus cosine inverse x they can happen to be the integral of same function further we

look at some more formulas as d by d x of tan inverse x we know that is one by one plus

x square and therefore integration of one by one plus x square d x is equals to tan x plus

constant and parallel to that previous d by d x of cot inverse x is minus one over one plus x

square which takes minus here and therefore d x over one plus x a squared is minus of say

is missed here cot inverse x plus the constant d by d x of sec inverse x is one over x square root of x

square minus 1 and therefore integration of x square root of x square minus 1 turns out to be

sec inverse x plus constant similarly d by d x of cos x inverse x with a negative sign equals to one

over x square root of x square minus one therefore integral d x over x square root of x square

minus one can also be written as minus of cosec inverse x plus constant c

so with kinematic

and inverse trigonometric function we have the relationship of logarithmic and exponential

function we have the knowledge of d by d x of e raise to power x as e raised to power x which

gives us the exponential of x integral d x same as e raised to power x in fact we can write that

d by d x of e raised to power n x divided by n as e raised to power n x for n bigger than

0

so that integral of e raised to power n x d x is same as e raised to power n x divided

by n plus a constant for n bigger than 0 or rather n naught equals 0 the same is true for n negative as well because as n becomes zero here this

function

will become one and we already know the integral for one d x and we also know the

d by d x for log of mod x that is one by x and therefore integral of one by x d x will

be written as log of mod x plus constant

so this ah case which which when we are discussing for x

raise to power n n naught equals to ah minus one

so you can understand that that the case when

n equals to minus one can be taken care of by this formula this formulas let me remark uh

that they are very important and since they are very fundamental

so we should remember them

because we will be using them very frequently one more important remark which i would like to

put here before i proceed with the example is that it may not be possible to find out the integral

of all functions in terms of elementary functions there may be some function

for which we may not know that what is its anti-derivative by inspection or even by evaluation one such example may be  $e$  raised to power minus  $x$  square  $d x$  so finding out that the anti derivative for this function in terms of elementary function that means that polynomial trigonometric investing number trick exponential etcetera is not possible so certain cases we may not be able to evaluate and in those cases we leave indefinite integrals in their own form as they are

so now we shall look at some examples depending on the properties and integrals which we have learnt the first example that i have chosen is very simple to find out the integral of the function for  $e$  raised to power  $3x$  plus  $1 dx$

$x$

so if you look at this integral first of all it is sum of two functions and therefore we use the property of distributive nature of integral on summation and write it in this form  $e$  raised to power  $3x$  taking constant  $4$  outside plus second integral  $1 dx$  now from the formula we know  $e$  raised to power  $n x$

so integral of  $e$  raised to power  $3x$   $3x dx$  turns out to be  $e$  raised to the power three  $x$  by three plus constant

so put it as integral  $e$  raised to power three  $x$  by three plus four times that constant

so we will call it as  $4c_1$  plus integral one this we already know that this is integral  $x$  plus the constant  $c_2$

so that the entire term becomes  $4$  by  $3 e$  raised to power  $3x$  plus  $x$  plus  $4c_1$  plus  $c_2$  since  $c_1$  and  $c_2$  both are constant

so we can club them together and we can rename them as a new constant so it will become  $4$  by  $3 e$  raised to power  $3x$  plus  $x$  plus a constant  $c$

so the integral turns out to be this much now here you you you can also do that that while integrating either you substitute the constant when you are integrating an integral or you can also do that you can substitute the constant at the end

so many times we do not we or we may not substitute the constants immediately while evaluating a certain integral rather we will be doing it at the end by plugging a single a single constant

so we will take another example for you say we are to evaluate integral of square root of  $x$  minus one by square root of  $x$  whole square  $d x$

so many times we may not have the application of integrals which we have learnt ah directly we may have to do some simplification for example here if you see if we expand the square what we will get is that square root square means  $x$  plus one by square root  $x$  square means one by  $x$  minus two times the product that is two apply the linearity property here

so what

we will get is that integral of  $x dx$  plus integral one by  $x dx$  minus two times

integral one  $dx$  which you can evaluate as here using the formula  $x^2$  by 2

plus 1 by  $x$  this is log of mod  $x$  minus  $2x$  plus a constant of integration therefore this

is the integral for this case

so a problem which may look initially a little bit complicated but if we use certain relationships which we already know this can be simplified and further we can figure

out that the integral will become very easy similar example another one which i will take for you is let us take  $x^3 - x^2 + x - 1$  divided by  $x - 1 dx$

so it looks a little complicated at the beginning

but if you look carefully you can figure out that in first two terms you can take  $x^2$  as common

so that will come  $x - 1$  plus the second term as  $x - 1$  whole divided by  $x - 1$ .

now

you see by dividing by  $x - 1$  we will get  $x^2 + 1$

so the complicated looking term

here is nothing but  $x^2 + 1$  for which now we can immediately figure out the integral  $x^2$

square and therefore it will be  $x^3$  by three one and therefore it is  $x$  and constant so

notice that that we have not distributed now this integral over summation we have

directly written it

so in due course of time with practice you can directly write the integrals and we will omit all these integral details when we are evaluating the

integral will put another example for you using some

trigonometric relationship example four say we have to evaluate  $\sec^2 x$  divided by  $\cos^2 x dx$

so we directly we do not have a formula

here but if you look at these carefully and apply the trigonometrical relationship

that  $\sec^2 x$  is nothing but one by  $\cos^2 x$  and  $\cos^2 x$  square  $x$

is nothing but one by  $\sin^2 x$

so we can write it as  $\sin^2 x$

over  $\cos^2 x$  which is nothing but  $\tan^2 x dx$  now again we do not know the integral

for  $\tan^2 x$  but we know relationship of  $\tan^2 x$  with  $\sec^2 x$  and

we know the integral of  $\sec^2 x$  so we have to think that what we know and how we can convert the problem into formula or a problem which we

already know

so we know that the formula one plus  $\tan^2 x$  equals to  $\sec^2 x$  and therefore using this formula here we can put it as  $\sec^2 x - 1 dx$

which will give you so

integral of  $\sec^2 x$  is  $\tan x$  minus integral of one is  $x$

plus constant of integral

so this little bit complicated looking formula

after doing certain calculation we reached to a relationship which we knew and

we used that relationship and we ultimately found the integral we will put a compression of differentiation and integration one is that that both of them are operators which operate on functions differential is also an operator and integral is also an operator operators take functions as input what i mean to say is that for example  $d$  by  $d x$  of  $f x$  so it is operated on the function  $f x$  then only it gives you  $f$  prime  $x$  and similarly here integral of  $f x d x$  it is operated on the function effect to give you a function  $f x$  so they are operator both of them satisfy linearity property integral also satisfy linearity property this we have seen differentiation if we take of a function it is unique so derivative of a function is unique integral we have seen if we take the integral of a function it is  $f x$  plus  $c$  so it is not unique in the sense of  $u$  the way we define uniqueness but we call it as most of the time unique up to a constant which means that if we ignore the constant then those integrals are unique you can define the derivative of a function at a point which means that it represents the direction of tangent at the point but no such meaning can be assigned in case of integral that means that that integral at a point has no meaning while differential at a point has a meaning of the direction of the tangent we also have seen the geometrical interpretation for the integral for the family of curves for the case of integral and similar geometrical interpretation for  $d y$  by  $d x$  is also understood we have seen for the case of derivative that it is a limiting process and same you will learn about integral as well is that it is limiting process finally as i have already mentioned for one property is that that integrals are considered as inverse operator of differentials but as i mentioned that that they are not essentially exactly the inverse operators because of the presence of the constant next we are going to learn how to evaluate integrals so there is no specific method which will be applied to each hand every function and depending on a function on a on or a particular problem we have to apply different methods so we will go through them one by one the first method which i am going to discuss for you today is method by substitution as you can see from the name substitution so what we do in this method is that in order to evaluate the integral  $f x d x$  we notice that that the independent variable here is  $x$  we change this variable independent variable  $x$  to another independent variable  $t$  by mean of some relationship say for

example assume that  $x$  is some function of  $t$  which has certain properties at this

differentiability

so that we can differentiate it then this will give us  $dx$  by  $dt$  is equals to  $g'(t)$  and

therefore in terms of differentials we can write it as  $dx$  is equals to  $g'(t) dt$

so the original

integral if i name it as  $i$  it turns out to be integral

of  $f$  replacing  $x$  by  $g(t)$   $dx$  by  $g'(t) dt$

so the formula for integral  $\int f(x) dx$  if i make

the change of independent variable from  $x$  to  $t$  it converts into another formula of integral

of  $f(g(t)) g'(t) dt$

so i will rewrite here integral of  $f(x) dx$  can be written

as integral of  $f(g(t)) g'(t) dt$  now we have already mentioned that that

these variables of integrals  $x$  and  $t$  they are dummy and therefore ah sometime it may also happen

that instead of choosing  $x$  as  $g(t)$  we may choose ah  $t$  as  $g(x)$  that means that  $t$  as a function of

$x$

so some certain function of  $x$  we can choose as  $t$  and then we can proceed with that ah

substitution that will be clear in due course of time i will take very simple example

so let

us take the example here that we are to find out the integral of  $\frac{2x}{1+x^2} dx$

so we cannot immediately get this integral by

the elementary formulas which we already know but if you notice that the denominator

term here if you differentiate it what you will get is  $2x$  which is same as the numerator term here and therefore if you look here carefully the

derivative multiplied by differential can be written

as differential in another variable

so if i think this function as  $g(x)$  then this is

nothing but  $g'(x) dx$  and therefore i can convert it into a new variable  $t$  let us see how we can do it

so define  $t$  equals to  $1+x^2$  or sometime we also say that substitute  $1+x^2$  equals to  $t$

so that  $dt$  the differential we always write it in

this fashion  $dt$  is equals to derivative that is  $2x$  times differential in  $x dx$

so  $dt$  equals to  $2x dx$  making this substitution in the given integral call this integral as  $i$  we shall get  $\int \frac{1}{t} dt$  and now this form is converted into the form

which we already know and this will give us  $\log |t| + \text{constant}$  but our problem was in  $x$

so we have to go back to

$x$  and therefore substitute for  $t$  make it as  $\log |1+x^2| + \text{constant}$

c

so this becomes our final integral for this case another simple example integration of sine of  $a x + b$   $dx$

so you can easily see if i take  $a x + b$  as some new variable  $t$  we know the integral of  $\sin t$  so to evaluate this integral we substitute  $a x + b$  equals to  $t$  so that  $a dx$  is equals to  $dt$  and the integral becomes  $\int \sin t dt$  by  $a$  which we will put here one by  $a \int \sin t dt$  so one by  $a$  integral of  $\sin t$  is nothing but minus of cosine  $t$  and finally we will add a constant  $c$  which will give you minus of  $\cos t$  is already known to us is as  $\frac{a x + b}{a} + c$  in fact this relationship can be generalized which we will see in our next class that if we are given a function which is having linear term as  $a x + b$  then it is always the integral of that function divided by the constant so we shall summarize whatever we have learnt today so we learnt the properties of indefinite integrals we also learnt some elementary formula we learnt how to evaluate simple integral we also learnt the completion of differentiation and integration and finally we learnt the very important method method of substitution thank you you