

today we are going to learn a new concept known as integration you have already seen the idea of differentiation

so integration in a sense can be considered as inverse process of differentiation the development of differentiation started with a problem of finding a tangent or a derivative to a curve for example suppose if we are given a function  $y$  equals to  $f(x)$  then at a point  $x_0, y_0$  if you want to find out the direction of the tangent you know that the  $\frac{dy}{dx}$  or the derivative of the function  $y$  is equals to  $f'(x)$  that helps in evaluation of the direction of the tangent

so  $\frac{dy}{dx}$  is same as slope of tangent this derivative has several applications which you have already seen ah during the course in differential calculus one of the example which i would like to cite is finding out the velocity suppose if you know the position of a particle at each time  $t$  then the derivative of the position function will give you the velocity of that particle the motivation of integration started with that how to find out area of different curves which are bounded by  $x$  axis though we ah study the differential calculus first then we go to the integral calculus but historically development of integral calculus that means the development of how to find out area of curves or certain structures that is started long before then ah that of the differential calculus two main mathematicians whom i would like to mention in this regard are lebanese and newton who have actually contributed for the development of present day calculus the notations which we used in in the present day calculus they are more close to those of lebanese

so now what we are going to learn today is about integrals broadly speaking we can categorize integrals into two types one is indefinite integrals and other one is definite integrals before i get into mathematical formulation of indefinite integral and definite integral i would like to pose a question why this topic

so in order to answer that question i will put couple of examples assume that a function of  $x$  which is defined on some close interval  $a, b$  is continuous and differentiable on open interval  $a, b$

so that  $f'(x)$  is known at each point of this interval  $a, b$  in case of differential calculus you used to find out given a function what will be the derivative of the function

so you differentiate the function and get the derivative of the function but here if i pose a question given  $f'(x)$  that means the derivative of the function is given to you can we find the function  $f(x)$

so you understood it clearly in case of differentiation we are given a function for which we have to find out the derivative but here we are given the derivative of the function and we need to know what the function will be let me pose another problem for you again assume that  $f(x)$  is continuous function on some interval let us say  $a, b$  such that if we assume that this be the graph of the function  $f(x)$  this will be the point  $x$  is equals to  $a$  and this be the point  $x$  is equals to  $b$

so on the interval what will be the area if i represent it by  $A$

so can we determine area which is bounded by this curve and the axis of  $x$  along with two lines ah  $x$  is equals to  $a$  and  $x$  is equals to  $b$

so these two problems basically if the function the derivative of the function is given then to find out the function or to find out the area of a continuous function bounded by the axis of  $x$  and some lines parallel to  $y$  axis these two problems fall in the category of integration the problem one closely relates itself to the class of indefinite integrals or you can say it may lead to that of indefinite integrals and the problem two which i post leads to that of definite integrals and together this formulates

so called integral calculus you may be thinking that indefinite integral and definite integrals are two different entity but basically they are interconnected though initially we will will understand and will study them separately as we develop the theory you will realize that they are very much closely interconnected

so to look at the connection we will start with the second problem of defining area function

so suppose that  $f(x)$  is  $x$  and it is given in interval  $0$  to  $a$  such that  $a$  is positive  $i$  have chosen this function

so that we can easily compute the area if  $i$  draw the graph of the function let  $a$  be any positive number

so that we can assume that  $a$  is here and  $x$  is equals to zero is here

so the function graph of  $f(x)$  is equals to  $x$  will look like this this is the point  $a$  comma  $a$  now what  $i$  want to know here is that that can  $i$  represent the area which is bounded by the curve and the axis of  $x$  as a function of the variable  $x$  such that each point of the interval  $0$  to  $a$  if  $i$  substitute that value  $i$  can get the value of the area that means  $i$  want to know a function  $A(x)$  where  $x$  is any general point which is lying between zero and  $a$

so if this be  $x$

so  $i$  want to evaluate the value of  $A(x)$  as the problem chosen is simple here the area shaded area of the shaded region is that of a triangle  $i$  can simply find it out as  $A(x)$  is equals to half of base which is the length  $x$  here multiplied by height since the function is  $y$  equals to  $x$  and therefore the height will be same as the base

so half of  $x$  into  $x$  which makes it half of  $x$  square

so the area function is given by half of  $x$  square if  $i$  want to know the value of the total area for the integral  $0$  to  $a$   $i$  can simply put here  $x$  is equals to small  $a$  and  $i$  will get it as half of  $a$  square also note that  $A(0)$  is zero and for any point in between  $i$  can use this formula to get the area

so therefore  $i$  have obtained a general formula for the area function in this case since the function was simple therefore  $i$  could use my simple tool of geometry that is the area of triangle to figure out the area but once this function becomes a general function or a complicated function it is it becomes little difficult to evaluate areas

so we obtain the area function which gives you the area of curve above axis of  $x$  now what information we can extract from here

so let us look at the area function closely and notice that  $d$  by  $d x$  of  $A(x)$  which is  $d$  by  $d x$  of  $x$  square by two in the previous example we obtain that  $A(x)$  is equals to  $x$  square it turns out to be two  $x$  by two which is nothing but  $x$  that means  $d$  by  $d x$  of area function equals to the  $x$

so what it is notable here is that that if we take the derivative of the area function we get the function itself the original function itself now if you look at the problem one which we post earlier says that that given  $f'(x)$  can be find  $f(x)$

so here if we are given  $f'(x)$  that means this  $x$  value can be find  $f(x)$

so  $A(x)$  in that case can be related to  $f(x)$  before moving that since this example  $i$  have posed ah with the help of a function  $y$  is equals to  $x$  let me introduce a theorem which is known as first fundamental theorem of calculus suppose  $f(x)$  be a continuous function on closed interval  $a$   $b$  and  $A(x)$  is the area function then the derivative of area function gives you the function

so the relationship which we observed for the case of  $y$  is equals to  $x$  function is in fact true for all functions and this result is known as first fundamental theorem of calculus next we shall look at the idea of anti derivative as we are aware from differential calculus that the derivative of certain functions can be

easily found

so we will use that idea and understanding of differential calculus to figure out if we can find out or if that can help us in finding out these integrals

so take the example  $\sin x$  we know that  $d/dx$  of  $\sin x$  is  $\cos x$  take another example of  $e$  raised to power  $n x$  by  $n$  this also we know that it is  $e$  raised to power  $n x$  also know the derivative of  $\tan x$  this is in fact  $\sec^2 x$

so if you look at this differentiation carefully what they tell is that the derivative of  $\sin x$  is  $\cos x$  derivative of  $e$  raised to the power  $n x$  by  $n$  is  $e$  raised to power  $n x$  derivative of  $\tan x$  is  $\sec^2 x$  the functions  $\sin x$  is known or will be called as anti derivative of  $\cos x$  and similarly the function  $e$  raised to the power  $n x$  by  $n$  will be called as anti derivative of  $e$  raised to power  $n x$  and function  $\tan x$  will be called as anti derivative of  $\sec^2 x$  therefore we define  $\sin x$  as anti derivative of  $\cos x$   $e$  raised to the power  $n x$  upon  $n$  as anti derivative  $e$  raised to power  $n x$  and  $\tan x$  as anti derivative of  $\sec^2 x$

so as i mentioned initially that integration or integral they can be considered in a sense as the inverse process of differentiation which can be visible from here is that the differentiation of  $\sin x$  is  $\cos x$  and similarly the anti-derivative for  $\sin x$  sorry  $\sin x$  is the anti-derivative of  $\cos x$  but as we know that if we find out  $d/dx$  of  $\sin x$  plus one what we shall get is that  $d/dx$  of  $\sin x$  plus  $d/dx$  of the constant one and we know that the constant differentiation of the constant is always zero and therefore it will turn out to be  $d/dx$  of  $\sin x$  which is nothing but which is nothing but  $\cos x$  therefore  $\cos x$  has  $\sin x$  plus one also as anti derivative

so what we saw earlier is  $\sin x$  is the anti derivative of  $\cos x$  now we saw that  $\sin x$  plus one is also anti derivative of  $\cos x$  and this in fact is true for all constant because we know that the derivative of constant is zero that  $d/dx$  of  $\sin x$  plus  $c$  is equals to  $\cos x$  and therefore  $\sin x$  plus  $c$  is anti derivative of  $\cos x$  where  $c$  is some constant we call it arbitrary constant taken to be a real number

so what we noticed is that given a function if we use the root of anti-derivative there can be infinitely many anti-derivatives by choosing constant  $c$  this in fact is true for general function  $f x$

so assume that  $d/dx$  of capital  $f x$  is equals to small  $f x$  then  $d/dx$  of  $f x$  plus  $c$  will also be equals to small  $f x$

so in general if  $f x$  is anti derivative of small  $f x$  then  $f x$  plus  $c$  will also be anti derivative of small  $f x$  in fact  $f x$  plus  $c$  such that  $c$  is a constant this represents set of all the anti derivative of effects or it may also be called as family of one parameter curves the the value of  $c$  which is obtained here is often very important and is dependent on the particular problem which we are handling which we shall see may be at some later stage next we define now formally integral in fact there is no distinction ah when we call integral or an anti-derivative they are same

so when we write integral there is a notion which is used

so as we defined  $f x$  plus  $c$  as set of all the anti derivatives of the function small  $f x$  we represent it in this way

so the set of all the anti derivatives or the for the function small  $f x$  is represented as the symbols  $\int$  which we call as integral symbol this term  $f x$  for which it is obtained this is called integrand this  $x$  with respect to which the function variable is evaluated is known as variable of integration  $f x$  capital  $f x$  is called integral or anti derivative  $c$  is known as arbitrary constant and this entire expression is known as the integral expression and we call it as indefinite integral

so an important remark here as i mentioned here the variable of integration  $x$  in fact is it is a dummy variable that means that this  $x$  can be replaced with any other variable say for example integration of  $f(t) dt$  is same as integration of  $f(x) dx$  that means it is immaterial whether you write  $t$  as the variable of integration or  $x$  as the variable of integration the result will be same what it is important here is that that which function you are evaluating

so if we take the previous examples which i showed you then in terms of integral representation we can write integral of  $\cos x dx$  is same as  $\sin x + c$  second example integral of  $e^{nx} dx$  i showed you that differentiation of  $e^{nx}$  upon  $n$  is  $e^{nx}$  and therefore integration of  $e^{nx}$  will be  $e^{nx}$  by  $n$  plus constant third example integration of  $\sec^2 x dx$

so since differentiation of  $\tan x$  gave you  $\sec^2 x$  therefore integration of  $\sec^2 x$  will give you  $\tan x$  the anti-tan sorry  $\tan x + c$  and fourth example we have seen integration of  $x dx$  is equals to  $\frac{x^2}{2} + c$  this we saw as an area function and we also saw the derivative of this area function is nothing but this function and therefore this function becomes the anti-derivative for the function  $x$

so these are the examples of these are some of the examples of indefinite integrals as i remarked the integration of  $\cos t dt$  will be  $\sin t + c$

so if the variable of integration is  $t$  instead of  $x$  it will give you the same function with new variable

so now we shall look at example of finding anti-derivative just by inspection we may call it as integral also

so first example which i am going to choose is  $f(x) = \sin 2x$  now look at this we are as we are aware that anti derivatives they come via the inverse process of the differentiation and therefore ah if i am getting sine function i must have differentiated a cosine function

so let us see what will happen if i differentiate a cosine function i will get sine function but then mind that that there is a term  $2x$  also and therefore instead of differentiating  $\cos x$  i must be differentiating  $\cos 2x$

so if i differentiate  $\cos 2x$  i am going to get twice of  $\sin 2x$  with a negative sign

so what i will do here is that i will put a negative sign here put a by two here what one can do is that that one can figure out the derivative of  $\cos 2x$  as minus of two  $\sin 2x$  and then the calculation will be like  $\frac{d}{dx}$  of  $-\frac{1}{2} \cos 2x$  is equals to  $\sin 2x$  and therefore this becomes anti derivative

so the anti-derivative in this case is  $-\frac{1}{2} \cos 2x + c$

so just observing the function and relating it with the differentiation or the derivative we can find out the integral or the anti derivative take another example say  $f(x) = e^{4x}$  we know that the differentiation of exponential function is another exponential function only difference here is that that this is exponential of  $4x$

so therefore we have to take care of that four taking as  $\frac{d}{dx}$  of  $e^{4x}$  by four equals to  $e^{4x}$  and hence the anti-derivative can be written as  $\frac{e^{4x}}{4} + c$  another example which may choose as  $f(x) = \sin 2x - 4e^{3x}$  now here look at this example this is having two functions signed to  $x$  and exponential function

so what we do here is that that we need as we know the distribution the the differentiation function can work on the linear combination of two functions and with previous two examples we can write that  $\frac{d}{dx}$  since the sine function is appearing this must be  $\frac{1}{2} \cos 2x$  this part we already know and minus  $4$

$e$  raised to the power  $3x$  we already know how do we compute for  $e$  raised to a  $4x$  so in the similar fashion we can go for this one will be  $e$  raised to power  $3x$  by three

so looking at this function we can simply figure out that what will be the actual anti-derivative its  $\frac{1}{3} \cos^2 x - \frac{4}{3} e^{3x}$  and plus a constant

so we have figure out that if given a simple function we can write the anti-derivative or the integral of that function what will happen in case of some complicated functions

so before getting into the examples further we will look at this another remark two which says that that if the derivative of two functions are same  $x$  belonging to some interval  $i$  then  $f(x) - g(x)$  is constant that means  $f(x)$  and  $g(x)$  they both belong to same family of curves

so its easy to see the proof for for this assume that  $h(x)$  is a function which can be represented as the difference of  $f(x) - g(x)$  take the derivative

so that  $h'(x) = f'(x) - g'(x)$  this for all  $x$

so we know that that  $f'(x)$  and  $g'(x)$  they are all same and therefore this becomes equals to zero  $h'(x) = 0$  for all  $x$  implies that that  $h(x)$  must be constant and hence the fact that  $f(x) - g(x)$  constant is established that means that the two functions they both belong to same family of curves this fact i will show you with help of another example

so consider  $\frac{d}{dx}$  of  $\sin^{-1} x$  which we know is that is square root of one minus  $x^2$  and  $\frac{d}{dx}$  of  $\cos^{-1} x$  this is minus one by square root of one minus  $x^2$

so these two are known results from the differential calculus we will use them by writing  $\frac{d}{dx}$  of  $\sin^{-1} x - \frac{d}{dx}$  of  $\cos^{-1} x$

so we see that the derivative of  $\sin^{-1} x$  and  $\cos^{-1} x$  are same and therefore the difference from the previous remark can be written as  $\sin^{-1} x - \cos^{-1} x$  will make it plus  $\cos^{-1} x$  a constant in fact this constant can be evaluated by putting  $x = 1$  in this expression

so that you will know  $\sin^{-1} 1 = \frac{\pi}{2}$   $\cos^{-1} 1 = 0$  that will make constant to be  $\frac{\pi}{2}$  and therefore this constant is nothing but  $\frac{\pi}{2}$   $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$  this is a famous identity for inverse trigonometric functions which you already know ah the claim is that that both  $\sin^{-1} x$  and  $\cos^{-1} x$  they belong to same family of curve one can see that by plotting simple graph here which i will do

so we know that the domain of these curves  $\sin^{-1} x$  and  $\cos^{-1} x$  is minus one to one

so plot  $\sin^{-1} x$  that takes the range from let us say this value is minus  $\frac{\pi}{2}$  this value be let us say  $\frac{\pi}{2}$  this value let us say  $\frac{\pi}{2}$  and similarly this value let us say  $\frac{\pi}{2}$

so for the case of sine function it goes somewhere from here like it should start from minus of  $\frac{\pi}{2}$  and go till  $\frac{\pi}{2}$  for the case of  $\cos^{-1} x$  you know for the range of minus one to one it should start from minus

so it should start from  $\frac{\pi}{2}$  and then go like this but since this function is your and this function is your but since the equivalence which we are claiming here is of  $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$  therefore instead of  $\cos^{-1} x$  we should look for  $\frac{\pi}{2} - \cos^{-1} x$  nothing but the mirror image of the function of  $\cos^{-1} x$  which will if you take the mirror image putting a mirror at the axis of  $x$

so if you notice this distance will be  $\frac{\pi}{2}$  ah this function will be minus of  $\cos^{-1} x$

so now you can clearly think that all the points they are minus  $\frac{\pi}{2}$

different

so the function sine inverse  $x$  and minus cosine inverse  $x$  they both both of them look similar that means they belong to same family of curve in fact we can also put a geometrical interpretation for the idea of anti-derivative of or integration for that consider the function  $y$  equals to  $e$  raised to power  $x$

so if you consider this function  $y$  is equals to  $e$  raised to the power  $x$  then we know that that for function  $e$  raised to power  $x$   $e$  raised to power  $x$  plus  $c$  is collection of all the anti derivative or it represents integral of  $e$  raised to the power  $x$   $dx$

so the the all the all the anti derivative of  $e$  raised to power  $x$  are given as  $e$  raised to the power  $x$  plus  $c$  how do they look like

so let us start with the value  $c$  is equals to  $0$

so that the first anti-derivative which you will get is  $e$  raised to power  $x$  assume that this is the point 1 this is 2

so this we can say that this  $x$  axis this  $y$  axis and therefore this is one unit and  $y$  axis the point is one ah zero comma one then zero comma two and then ah

so on and

so forth

so this is three in that sense four and similarly here this is zero is minus one and then

so on and

so forth

so if you plot  $e$  raised to power  $x$  you know that if you put  $x$  is equals to zero here you will get one therefore one point is here further you can plot some other values by putting some some more values say if you put  $x$  is equals to one it goes to  $e$  and you know for let us say this is the point  $x$  is equals to 1

so  $e$  you know value is 2.

7 therefore it will be somewhere here

so in between it should go smoothly like this and similarly other values you can plot and as  $x$  goes to negative large value this value goes to zero

so  $x$  axis becomes a tangent to the curve as  $x$  goes larger in the negative  $x$  axis

so this is the curve  $e$  raised to power  $x$  similarly if i put here  $c$  is equals to one the next curve i will get is  $e$  raised to the power  $x$  plus one

so the next curve  $e$  raised to power  $x$  plus one how shall i plot  $e$  raise to power  $x$  plus one again if i put  $x$  is equals to zero here what i get is two

so that means that the point of intersection with  $y$  axis is two and since the two curves are parallel

so therefore this and in this case the line  $y$  is equals to ah one that will be tangential to the curve  $e$  raise to power  $x$  plus one similarly the other curve i can plot for you is this way

so this is  $e$  raised to power  $x$  plus two  $e$  raised to power  $x$  plus one

so now you can in general plot other curves also going negative direction

so this curve becomes  $e$  raised to power  $x$  minus one this becomes  $e$  raise to power  $x$  minus two

so we can get all the curves by sliding  $e$  raised to the power  $x$  curve upward direction or downward direction depending on whether constant  $c$  is a positive constant or a negative constant now look carefully at the point of intersection with  $y$  axis let us rename these points as  $p$  naught  $p$  1  $p$  2  $p$  3 and similarly

so on and

so forth and if we evaluate the  $dy$  by  $dx$  that is the derivative at the point let us say starting with  $p$  naught will be  $dy$  by  $dx$  of  $e$  raised to power  $x$  sorry  $p$  naught is  $e$  raised to the power  $x$  minus one  $p$  naught is  $e$  raised to power  $x$  minus one then you will get  $e$  raised to power  $x$  at  $p$  naught at  $p$  naught  $x$  is

zero and therefore you will get the value one similarly you evaluate  $\frac{dy}{dx}$  at  $x=1$

so at  $x=1$   $y=1$  is at this point

so this is corresponding to curve  $y = e^x$  therefore you will get  $\frac{dy}{dx}$  of  $e^x$  at  $x=1$  equals to  $e^1$  at  $x=2$  that will be corresponding to  $e^{x+1}$  and that will also turn out to be same

so what is what i am trying to point out here is that the direction of tangent at each point of intersection of  $y$  axis with each member of this family of curves is same as one in fact if you draw a line parallel to  $y$  axis say  $x = c$  and evaluate the tangents at those points then you will realize that the  $\frac{dy}{dx}$  corresponding to let me call this points as corresponding to  $e^{x-1}$  i will call it as  $x=c$  corresponding to  $e^x$  curve i will call as  $x=c+1$  and corresponding to this curve i will call this  $x=c+2$  similar to  $\frac{dy}{dx}$  at  $x=c$  will come from  $\frac{dy}{dx}$  of  $e^{x-1}$  at  $x=c$  which will be equals to  $e^c$  and the point  $x=c$  has this  $x$  value as 1 and therefore this can immediately be put as  $e^1$  which is  $e$  and similarly the case of  $x=c+1$  you can evaluate  $\frac{dy}{dx}$  at  $x=c+1$  will be coming from  $e^x$  derivative of  $e^x$  will be  $e^x$  again since  $x$  value is same therefore you will get the value  $e$

so the direction of tangent here at each of the point you will find that the slope is  $e$  similarly if you want you can find out the value at  $x=c$  minus one and you will find that the tangential direction at each of this point is nothing but  $e^{x-1}$

so what geometrically it interprets is that for a function if you get the family of curves then if you plot the family of curves and draw lines parallel to the vertical axis which is generally  $y$  axis then tangent at the intersection point of that vertical line with each member of the family will be same

so finally we look at remark three the case of area problem which we considered that can now in the symbol of integral representation can be written as  $\int_0^x x dx$  which we obtained as  $\frac{x^2}{2}$

so this is defined as definite integral if you notice here there are two values which are written as zero and  $x$  these are known as lower and upper limits which you will learn later half of this course

so i would summarize what we have done today is that we understood the definition of integrals we also understood what is the idea of derivative or integrals and finally we looked at what is the graphical representation of these integrals are family of curves

so using these basics in next class we will try to figure out i will try to understand how to find out the integrals of certain functions we will develop certain formulas and will use them to find out the integrals of some simpler functions and then to some other complicated functions thank you