

welcome students in this lecture we will consider miscellaneous exercises as an application of definite integrals and we will solve various problems and compute area of simple and complicated shapes let us consider the example that we have seen in the last class here it will be good if you assume  $a$  positive and  $m$  positive so that we can plot both the curves given so if this is your  $x$  axis and this is your  $y$  axis so  $y^2$  equals to  $4 - ax$  is a parabola whose vertex is  $(0, 0)$  and  $y$  equals to  $mx$  is a line passes through origin so if  $a$  is positive the parabola will be drawn like this and  $y$  equals to  $mx$  if  $m$  is positive you can plot like this the area bounded between them is this let us divide this area into elementary areas thin vertical strips so if one such strip is having  $dx$  as width so elementary area will be  $y$  from this minus  $y$  from this so we get  $\sqrt{4 - ax} - mx$  as this is our elementary area so required area is  $\int_0^{\frac{4}{a}} (\sqrt{4 - ax} - mx) dx$  where we are supposed to put the values of  $x$  so value of  $x$  starts from origin and it goes to this point to get the  $x$  coordinate of this point we need to solve both the equations so if you put  $y$  equals to  $mx$  here you get  $m^2 x^2 = 4 - ax$  so  $m^2 x^2 + ax - 4 = 0$  so  $x$  is  $\frac{-a \pm \sqrt{a^2 + 16m^2}}{2m^2}$  so this gives you  $x = \frac{4}{a}$  gives you 1 point of intersection that is origin and  $x = \frac{4}{a}$  by  $m^2$  gives you another point of intersection that is so this is your  $x = \frac{4}{a}$  by  $m^2$  so limits of integration will be  $x = 0$  to  $x = \frac{4}{a}$  by  $m^2$  so your required area is  $\int_0^{\frac{4}{a}} (\sqrt{4 - ax} - mx) dx$  now you can integrate this these are simple integrands where anti derivative is known to you so you get this  $\frac{2}{3} (4 - ax)^{3/2} - \frac{m}{2} x^2$  from  $0$  to  $\frac{4}{a}$  by  $m^2$  you get  $\frac{2}{3} (4 - a \cdot \frac{4}{a})^{3/2} - \frac{m}{2} (\frac{4}{a})^2$  so we get  $\frac{2}{3} (0)^{3/2} - \frac{m}{2} \frac{16}{a^2}$  minus  $\frac{2}{3} (4)^{3/2} - \frac{m}{2} (0)$  so we get  $-\frac{8m}{a^2} - \frac{16\sqrt{2}}{3}$  so we get  $\frac{16\sqrt{2}}{3} - \frac{8m}{a^2}$  is a final answer let us take another example compute area bounded between  $x$  axis  $y = 2x^3 - x^2 - 2x$  and  $x = -1$  to  $x = 2$ . now it seems that it is difficult to plot this curve  $y = 2x^3 - x^2 - 2x$  but unless you plot it you cannot find out the actual area because if you try to integrate it directly this will not give you a correct value of the

required area because you do not know where  $y$  is positive and negative between  $-1$  and  $2$

so this is

not the right approach

so you have to have some idea of the curve that where it is positive and where it is negative

so for that let us try to factor  $y$  equals to twice of  $x$  minus  $x$  square minus  $2x$

so you can write this as which you can write as

so you can see that  $x$  equals to zero

$x$  equals to minus one and  $x$  equals to two this curve is crossing  $x$  axis

so let us try to

plot it roughly

so that we have some idea where this where is area is above  $x$  axis and where is this area is negat below  $x$  axis let us plot

so that we can know where this area

is above  $x$  axis and where this area is below  $x$  axis since it vanishes at  $x$  equals

to  $0$  and minus  $1$  and  $2$

so let us draw these points minus one zero and two

so at these points curve is taking zero value that means it is crossing  $x$  axis now when  $x$  is between minus  $1$  and  $0$   $x$  plus  $1$  will be positive  $x$  will be negative and  $x$  minus two will be negative

so  $y$  is positive

so one once  $x$  is less than zero and greater

than minus one this term is positive this term is positive this is negative this is negative

so total is positive

so it will be like this similarly if  $x$  is positive

so  $x$  plus  $1$  is

positive but  $x$  plus  $x$  minus  $2$  will be negative when  $x$  is less than  $2$ .

so  $y$  is

negative when  $x$  is between zero and two

so the curve will be something like

this roughly

so the total area required is now shaded by green color you can see

so we have to compute both and add them together to get the actual area

so let me draw it again

so it goes like this and this your this equation of the curve is twice of  $x$  cube minus  $x$  square minus two  $x$

so two  $x$  cube minus  $x$  square minus  $2x$

so area say this is a  $1$  this is a  $2$  so

let us compute a  $1$  a  $1$  is minus  $1$  to  $0$   $2x$  cube minus  $x$  square minus two  $x$  so elementary area will

so this is a one elementary area will be this into

$d x$  and integrate it

so you get

so zero minus one by four then minus ok at zero it will be zero

so i have to put minus

of then mine

so this value

so this is plus and then this will be one by three and then this will be minus one

so this is minus two by seven and this is seven by twelve minus one sorry minus 2 7 by 12 minus 1

so we get minus 2 minus 5 by 12.

ok fine

so we get 5 by 6 this is correct so

then a 2 this is equal to 0 to 2 twice of  $x$  cube minus  $x$  square minus  $2x$  dx this

is equal to  $2x$  to the power 4 by 4 minus  $x$  cube by 3 minus  $x$  square 0 to 2 at 0 it

will be 0

so value will that will get is 2 16 by 4 is 4 minus 8 by 3 minus 4 so we get minus 16 by 3

so total area is a one plus mod a two

so five by six plus sixteen by three which is equal to thirty seven by six let us take another example find out area bounded between  $y$  equals to two minus  $x$  square and  $y$  equals to minus  $x$

so at the very first step for every problem on area you have to have some idea about the region where it lies where whether it is on positive side

or whether it is negative side

so plotting is must

so this represents a parabola inverted parabola whose vertex is 0 comma 2 and axis is  $y$  axis

so you will get this parabola now  $y$  equals to minus  $x$  is a straight line with the gradient minus 1

so you get this  $y$  equals to minus  $x$  and this is  $y$  equals to two minus  $x$  square

so area

bounded between them will be this

so again

so by the process similar to the previous cases divide this area into vertical thin vertical strips or rectangles that

is known as elementary area

so this elementary area will be two minus  $x$  square minus of minus  $x$  into  $dx$  minus  $x$  into  $dx$  you remember the formula where we have taken

two curves  $f(x)$   $g(x)$  says that  $f(x)$  dominates  $g(x)$  and the area bounded between them try to

recall

so we got  $f(x)$  minus  $g(x)$  into  $dx$  so  $f(x)$  is 2 minus  $x$  square  $g(x)$  is minus  $x$  so we got

this this is your elementary area now value of  $x$  for which the entire shaded area is drawn will be one value

is this this is  $x$  minimum and this is  $x$  maximum we need to find out both the values

and then we will put it here  $x$  minimum  $x$  maximum

so to to find out these  $x$  minimum and  $x$  maximum

values we need to solve equation of both the curves minus  $x$  is equal to 2 minus  $x$  square so

we get  $x$  square minus  $x$  minus two equals to zero

so values of  $x$  will be that will give you 2 comma minus 1

so this  $x$  minimum is minus 1 and this is two  
so this

entire shaded area lies between  $x$  equals to minus one and  $x$  equals to plus two

so we need to put the limits over here and we get two minus  $x$  square plus  $x$  dx  
integrated we get  $2x$  minus  $x$  cube by 3 plus  $x$  square by 2 minus 1 to 2

so we get upper limit value of

the anti derivative at upper limit is four plus two minus minus two minus plus one by three plus one by two

so this is equal to

so this 4 this and this this is  $8$  minus  $8$  by 3 minus  $5$  by this is eight hmm

so this is your final answer we are

solving miscellaneous problems

so that you are able to plot a curve then identify

the region and then you should be able to put the limits properly

so let us take

another example in the sequence area bounded between  $y$  equals to  $x^4$  minus  $x$  square  $x$  axis and  $x$  equals to minus two and  $x$  equals to plus two

so at the very first instance it seems that since

the area is bounded between minus two and two

so if you integrate  $x^4$  minus  $x$  square dx

from minus two to two you will get the value of the integral but what will happen if you do that

without plotting if you try to approach and plot and integrate this function from minus two to two so

this is an odd function

so immediately the value will be zero

so therefore there is some problem

with our approach

so without plotting we cannot proceed

so we should have some idea rough

idea the curve

so let us draw it first

so what you can see is that  $y$  is

$0$  when  $x$  is  $0$  and  $y$  is  $0$  when  $x$  is plus minus 2 now between between  $0$  and  $2$   $y$  is positive and between minus 2 and  $0$   $y$  is

negative

so it vanishes at  $0$  2 and minus 2 the curve vanishes

here here here and from minus 2 to  $0$  it is negative and from  $0$  to

2 it is positive

so we can have this kind of shape because it is continuous also to find out how

this curve will look like here and here exactly if you are interested in knowing more about a

curve you have to find out the derivative and find out the value of the derivative at  $0$  minus

2 plus 2

so you

so that you know how the curve is crossing at minus two zero and two

so actual

area that we are looking for is that is lying between the curve  $x$  axis and lines  $x$  equals

to minus two and  $x$  equals to plus two is this

so total area required will be say this is a

one and this is a two

so total area required is mod of a one plus mod plus a two

so we will

compute a two first

so a two will be given by zero to two  $x$  under root four minus  $x$  square  $dx$   
since this curve is symmetric and an odd

function of  $x$  this this a one will be having same magnitude as that of a two  
but negative sign will be there

so a 1 will be minus of a 2 and a 2 is 0 to 2  $x$  under root four minus  $x$  square  
 $dx$  now assume that  $x$  square is  $t$

so two  $x dx$  is  $dt$

so zero  $x$  zero  $t$  equals to zero and  $x$  two

is  $t$  is four

so limits will be this and  $x dx$  is one by two  $dt$

so you get one by two four minus  $t dt$  one by two integration will be zero to  
four four by three at four it will be zero

and at zero it will be zero minus ah minus sign will appears as you have minus  
 $t$  here

so at four it

will be zero and then at zero it will be four to the power three by two minus  $i$

so you

get minus plus four by three into eight that is sorry we have one mistake  
here we get

here we get one by three

so you get one by three

so finally we get 8 by 3

so  $a^2$  is 8 by 3 a one will be minus eight  
by three because function is odd

so total area required is sixteen by three let us take another example area  
bounded between two parabolas  $y$  equals to  $x$  square and  $y$   
equals to eight minus  $x$  square

so let us plot them  $y$  axis  $x$  axis  $y$  equals to  $x$  square is

parabola vertex 0 0 and axis is  $y$  axis we get this and  $y$  equals to 8 minus  $x$   
square

is parabola which is inverted at whose vertex is 0 comma eight

so we will get this shape

so we need to find out these

two point of intersection and the required area is shaded by green color

so again we divide the area into elementary areas thin strips or rectangles  
if once such strip is

having width  $dx$

so elementary area will be eight minus  $x$  square minus  $x$  square  $dx$  we need to  
find out these two values to so

for that we need to solve both the equations

so this is minus 2 0 and this is 2 comma 0

so limit of  $x$  will be from minus two to two and minus two to two eight minus two  
 $x$  square  $dx$  since this is if you replace  $x$  by minus  $x$  you can see that  
the sign of this integrand will not change and  $f$  of minus  $x$  is equal to  $f x$   
therefore

it is an even function

so you can use the property of the definite integral that

minus  $a$  to  $a$   $f x dx$  is twice of 0 to  $a$   $f x dx$

so therefore we can write this as 2 0 to 2 8

minus two  $x$  square  $dx$  this is equal to two eight  $x$  minus two by three  $x$  cube 0

to 2

so we get this well

of the required area is  $2\sqrt{16}$  minus sixty four by three let us take another example find out area between  $y$  equals to one minus  $\cos x$  to  $\sin x$   $x$  axis  $x$  equals to  $0$  and  $x$  equals to  $\pi$

so this seems very complicated curve and it will not be easy to plot it but you should know whether it is positive or negative and you

should have some rough idea about the curve

so if this is  $x$  axis this is  $y$  axis

so between zero and  $\pi$  what is the nature of the curve one minus  $\cos x$  into  $\sin x$

so if say this is  $0$  this is  $\pi$

so at  $x$  equals to  $0$   $y = 0$  because of both the factors  $x$  is equals to  $\pi$  also  $y = 0$  because  $\sin \pi = 0$  now between zero and  $\pi$  is there any chance of

vanishing the function again or is there any chance that  $y$  takes negative value since  $\cos x$  lies between minus 1 and 1  $\sin x$  is as always positive or  $0$  between zero and  $\pi$  and one minus  $\cos x$  is always between zero and  $\pi$  therefore this curve is always positive

so you will have this kind of shape of the curve

so your required area is this and if  $dx$  is a width of elementary area then elementary elementary

area will be  $(1 - \cos x) \sin x$  into  $dx$  and this is your elementary area and total area will be  $\int_0^\pi (1 - \cos x) \sin x \, dx$  let us evaluate it  $\int_0^\pi \sin x - \sin x \cos x \, dx$

which you can write as zero to  $\pi$   $\sin x - \frac{1}{2} \sin 2x$

$\int_0^\pi (-\cos x + \frac{1}{2} \cos 2x) \, dx$

so this will give you minus one minus one and then at  $2\pi$  zero

you will get one minus one

so plus one by four one minus one

so finally you get  $\frac{1}{2}$  this problem is very simple but you

should remember that unless you have some idea about the curve whether it is on either

side of  $x$  axis or it is changing its sign you cannot find out the correct area let us take another example area bounded between  $y$  equals to  $\cos^2 x$  and  $y$  equals to one  $x$  equals

to zero and  $x$  equals to  $\pi$

so how to plot this region you know how the the graph of  $\cos x$  looks like between zero and  $\pi$

so at zero it is one then  $\pi/2$  it is zero and then it is again one at  $\pi$  it

is

so this is plus one this is minus one this is a graph of  $\cos x$  now if you square

it how the graph of  $\cos^2 x$  will look like

so this portion will go up

because you are squaring it

so this is

so this is rough sketch of  $\cos^2 x$

so and because the values lie between minus one and one

so it will not

cross one between zero and  $\pi$

so it will and pi by two it will be zero

so it will be something like this

so your area that you are looking for that lies

between the curve  $y = \cos^2 x$  and  $y = 1$  this is your required area now

why this is smooth here

so that is very

easy to find that why this  $\cos^2 x$  will be smooth at  $\pi/2$ .

so you need to just find out

the derivative and you can see that  $y'$  will be  $2 \cos x \cdot (-\sin x)$

and that will be

0 at  $\pi/2$ .

so therefore the curve is smooth the tangent at  $\pi/2$  is x axis

so required area

is therefore 0 to  $\pi$  elementary area will be vertical strips if we take

length  $dx$  then we need to apply

again that theory where we have two functions  $f(x)$  and  $g(x)$  and realize between  $a$  and  $b$

so you

write  $\int_a^b (f(x) - g(x)) dx$

so  $f(x)$  is one and  $g(x)$  is  $\cos^2 x$

so you get this this is equal to  $\int_0^\pi (1 - \cos^2 x) dx$  which is

one minus  $\frac{\sin 2x}{2}$  from 0 to  $\pi$  area therefore

is equal to  $\frac{1}{2} \times \pi - \frac{1}{4} \times 0 = \frac{\pi}{2}$

so you get  $\pi/2$

minus zero minus zero

so this is  $\pi/2$

so till now we have solved lot of examples on applications of definite integrals let us see there are certain

properties which we have not covered till now and let us discuss them

in brief so let us see what can be done in this regard a definite integral is written in this form where we have taken  $f(x)$  to be continuous on closed interval  $a, b$  and

our both end points of the interval are finite

so whatever problems we have solved  $f(x)$  was a continuous function

and both  $a$  and  $b$  are finite

so so there are two questions that

can be raised what if  $f(x)$  is discontinuous on  $a, b$  and another question is what if interval of integration is not bounded that is your interval is  $a$  to infinity

minus

infinity to  $a$  or minus infinity to infinity let us see how to deal with such situations for that

let us take the case where  $f(x)$  is discontinuous on the interval  $a, b$  but  $f(x)$  is piecewise continuous piecewise continuous

so to understand

this let us take integral from  $-\infty$  to  $\infty$  where this is known as greatest integer function and the plot of this the plot of this can be done as follows

so so greatest integer function is defined

like that between zero and one it takes value zero at one it takes value one

and between one and two just before two it takes value one throughout this is one and between minus one and zero it takes minus one at minus one it is minus one and between

minus one and two it is minus two sorry not this but somewhere here this is minus one this is minus two

so if you want to integrate this in this interval you can write this integral as equal to minus 1 to my minus 2 to minus 1 where the function value is minus 2 plus from minus 1 to 0 function value is minus 1 plus 0 to 1 function value is 0 plus 1 to 2 function is 1 dx if you integrate this you get minus 2 x minus 2 to minus 1 plus minus one integral is

x minus one to zero plus zero plus one two two sorry x one to two

so value is plus two minus one we are applying the formula of definite integral fundamental theorem of calculus two

so we get minus two here we get minus one and then one

so value is minus two

so now let us try to understand

so the

function is not continuous throughout but between minus two and minus one it is continuous between minus one and zero discontinuous from zero to one it is continuous

and from one to two it is continuous

so you can break the in uh integral in ah you can break the integral over those sub intervals and you can then evaluate it into on into ah on

individual integ on individual intervals now if i ask you to find out area under the given curve from minus 2 to 2

so area will be represented by this this and this then ah we have to take so area of

this was minus two area area of this will be ah mod of minus two that is two and area of

this will be mod of minus one that is one and here it is one

so if i ask you to

find out area then you have to put mod over here here because this is giving you negative values since the curve is lying below x axis

so to

put mod and then add it together to get the value of the area

so we have seen how to

evaluate a piecewise continuous function now let us observe what will happen if your

interval is not bounded or it is it contains plus infinity or minus infinity so there are

several cases in that

so let  $f(x)$  be continuous but interval of integration is not bounded

so the first category is say for example

so you can see that this

function is continuous everywhere but the interval of integration is infinite from 0 to infinity then another case is from minus to infinity to infinity one by one plus

$x^2$  then another case where function is discontinuous but interval is finite say a to b where a and b both are

finite here discontinuity does not mean piecewise discontinuity or piecewise

continuity

so this discontinuous function means it has infinite values somewhere in the interval of integration

so for example  $\int_0^1 \frac{1}{\sqrt{x}} dx$  by root x if you see the graph of  $\frac{1}{\sqrt{x}}$  it tends to infinity when x tends to 0 and it tends to 0 when x tends to

infinity

so this is the graph of  $\frac{1}{\sqrt{x}}$  this is y axis this is x axis

so i

want to compute the area of this area then third case is where the function is also

discontinuous and interval is also not finite so case three where function is discontinuous which are not which are not continuous also interval of integration is infinite for example minus infinity to infinity  $\int \frac{1}{x^2} dx$

so here  $\frac{1}{x^2}$  is not continuous at  $x = 0$  and intervals interval of integration is also unbounded unbounded or infinite

so let us solve some

examples and see how will you deal with such cases

so example one zero to infinity  $\int \frac{1}{1+x^2} dx$

so if you plot this curve at 0 it is 1 and at infinity it is 0 it will go like this

so this is the area that is governed by this integral

so if you see you cannot apply limit of finite sums because this you have to add infinitely many areas of elementary rectangles and

so the theory that is that has been developed

so far is for finite interval

so what we do there is a very simple trick what we do we just write  $\int_a^1 \frac{1}{1+x^2} dx$  and then since now a is finite

so we have

taken say some value here a and then we are integrating this area finally we take

limit of a tends to infinity and then the final value of the integral is computed

and theory for this is well developed

so so if the interval is finite

and this function is continuous then by fundamental theorem of calculus two we can write the value of the integral as

so  $\tan^{-1} x$  is the anti-derivative

so we get  $\tan^{-1} \infty$  that is  $\frac{\pi}{2}$

so what we have done this

infinity is dealt by converting this integral into limit of this integral which is defined over finite interval hence the problem of unbounded intervals can

be dealt in this manner take one more example  $\int_0^1 \frac{1}{\sqrt{x}} dx$

if you plot this curve you get  $\frac{1}{\sqrt{x}}$  one by root x tends to infinity this y axis this is x axis

so you want to compute this is one

so this  
 is  $x$  equals to zero this is  $x$  equals to one  
 so you want to compute this area  
 so you can  
 see that since the if you draw it in very small rectangles you can see that  
 when you move towards  
 $0$  you will not be able to write the area of the rectangles because function  
 value tends to  
 infinity  
 so what we do the idea is same again we assume that ok say this is  $\epsilon$   
 $\epsilon$  and  
 then we evaluate this area and then we take limit of  $\epsilon$  as  $\epsilon$  tends  
 to zero  
 so for that we need to write this  
 integral as  $\int_{\epsilon}^1 dx$  over  $\sqrt{x}$   $\epsilon$  tends to  $0$   
 so this is  
 equal to limit  $\epsilon$  tends to  $0$  now one by  $\sqrt{x}$  is continuous  
 in  $\epsilon$  close interval  $\epsilon$  to one  
 so you can apply fundamental  
 theorem of calculus and then you can write the value from  $\epsilon$  to one which  
 gives you  $2 - 2\epsilon$   
 so the value of the integral is  $2\sqrt{x}$   
 so you get  $2 - 2\sqrt{\epsilon}$  and after the taking the limit  
 you see that the value of the integral is  $2$  such integrals are known as improper  
 integrals such integrals you can deal with define  
 by defining the integral as a limit of integral over finite interval we have  
 seen  
 certain miscellaneous examples on definite integrals on area and definite  
 other kind of  
 definite integrals  
 so in our next class we shall continue with miscellaneous examples and see  
 how  
 to approach such complicated problems thank you you