

we are learning definite integrals and till now what we have learned that there are two methods to find out value of a definite integral one is by limit of sums and one is by using ant derivatives we have seen that there are definite integrals whose integrand is very simple so that anti derivative of small $f(x)$ exists and therefore you can write the value in this form so but there are several integrals where $f(x)$ is very complicated in that case say so this is very very complicated function whose anti derivative is difficult to find compute anti-derivative i find out $f'(x)$ says that this is equal to small $f(x)$ then what we have seen that we we apply in these cases we apply method of substitution method of substitution and what we do in this that we transform the integral where we will have new limits these limits will be different and we get another function which is simple enough to compute compute anti derivative that is there exist $g(t)$ such that there exist $g(t)$ such that this is equal to small $g(t)$ then we have also seen that there are seven eight properties of properties of definite integral if you use this properties of definite integrals and evaluate the integral life becomes much simpler and you can compute the integral much easier than otherwise so you are supposed to understand the properties prove it and then try to apply it on definite integrals to evaluate complicated definite integrals then there are lot of applications of definite integrals and as an application we have a started computing area of certain shapes and we started with very simple shapes like we have computed area of circles ellipses and we have also computed area of a curve bounded between the curve and a given line in the sequence we proceed further and solve some more problems to understand the definite integral in a much better way so let us take another example so find out smaller area bounded between ellipse and vertical line x equals to a where a is greater than b so let us plot ellipse and this line if this is your x axis and this is your y axis then since a is greater than b so so let us say that this point is $(-a, 0)$ and this point is $(a, 0)$ and this is $(0, b)$ and this is $(0, -b)$ then you can plot the ellipse like this and you know that the a is coordinate of the focus one of the focus of ellipse so say this point is $(a, 0)$ so the vertical line that is given to you is this so since we are looking for a smaller area bounded between the ellipse and this line so the required area is this again you know that ellipse is symmetric both about x and y axis because powers are even so it is symmetrical about x and y axis therefore this area is symmetrical about x axis so required area required area is area shaded by green color say this is a

so required area is equal to twice into area of shaded by green color this is equal to two a so now if you consider a vertical strip here with whose width is dx and height will be governed by the equation of the ellipse

so elementary area will be $y dx$ and if you integrate it from x from minimum to

maximum that is x equals to a to x equals to a you get the required area

so if you solve the equation for y you get two values of y that is one minus x square by a square under root plus minus b since

we are using upper portion the portion of the lift which is lying above x axis therefore will be using this equation a square

so therefore y is b by a under root a square minus x square

so required area is therefore $2 \int_a^a b \sqrt{a^2 - x^2} dx$ let us solve this integral and

get the value we get $2 \int_a^a b \sqrt{a^2 - x^2} dx$

so we can write this is a well known integrand and its value is known to you

so you can write the value directly

so value of this integral will be limit goes from a to a

so $2 \int_a^a b \sqrt{a^2 - x^2} dx$

it will give you zero

so value of the value of this function at a minus value of this function at a

so at eight it will be zero plus one by two a square sine inverse one minus half a under root a square minus a

square e square plus one by two a square sine inverse e

so we get $2 \int_a^a b \sqrt{a^2 - x^2} dx$

so we get πa^2 by four minus one by two a e this value you can

replace by b because you know that b^2 square equals to a^2 square minus a^2 square e^2 square for ellipse sorry there will be a negative sign over here please correct it

so we get minus

$\frac{1}{2} \pi a^2$ sine inverse e

so this is the answer of the problem that is given to you

so let us take another example and see how to compute area

bounded between two curves for that let us say case one assume that

$f(x)$ is greater than equal to $g(x)$ for all x in the interval a to b and they coincide at the point a and b

so if this

is your y axis this is your x axis

so you can plot of $f(x)$ and $g(x)$ in like this where

this point is say a and this point is b

so you assume that $f(x)$ and $g(x)$ they have same

value at a and b but $f(x)$ dominates $g(x)$ but $f(x)$ dominates $g(x)$ in the interval a to b

so a area

bounded between these two curves is this now how to compute this area

so divide this area into several very thin rectangles are known that

is also referred as elementary area and see if you just extend it to here on x axis

say this is dx

so area of this elementary strip will be this height into dx and this height will be this height will be $f(x) - g(x)$

so height is this into dx gives you area of this strip now if you integrate it

so this is your elementary area for this area for this required area this is $f(x) - g(x)$

so if you integrate it from $x = a$ to $x = b$

you get your required area we can understand this example in a different way also and derive this formula for that

let us draw the figure again and $f(x)$ this is $g(x)$ this is x axis y axis

so first let us compute this area

so value of this area will be

if you take this vertical strip you get $f(x) - g(x)$ this is a and b

so you get this gives you

this area shaded by green color now if you take this vertical strip which starts from zero and ends on $g(x)$ every time

so if you just

integrate this you get this area

so if you take $g(x) - f(x)$ is area of this strip this

is strip and if you integrate from a to b you get this red area shaded by red color and a

to b $f(x) - g(x)$ is area shaded by green color

so if you subtract them you get the required formula

so we have computed first green shaded area which is given by this

integral and then we have computed red shaded area which is given by this and if we

subtract we get the required area the required area shaded by blue and

so we get this formula finally let us

take another case case two where $f(x)$ dominates $g(x)$ when x belongs to a to c a close interval

and $g(x)$ dominates $f(x)$ when x belongs to c to b and the situation can be represented graphically as following

so we have $f(x)$ and $g(x)$ this is point c this is a and

this is b and we are looking for this area this curve is $f(x)$ this is $g(x)$

so let me draw it by different color

so that it is

clear to you

so from a to c $f(x)$ dominates and from c to b $g(x)$ dominates in this case by applying

the previous formula we can get required area as $\int_a^c (f(x) - g(x)) dx + \int_c^b (g(x) - f(x)) dx$ for red shaded area elementary area is this because $f(x)$ dominates

so elementary

area that is area of elementary strip or thin rectangle is $(f(x) - g(x)) dx$ and

for green shaded area elementary strip is this and elementary area will be given by $(g(x) - f(x)) dx$

now let us apply this formula and solve some problems

so find out area bounded between $y = x^2$ and $y = 4x$

let us draw the curves

so let us see what region we are getting for that you

know that $y = x^2$ is the parabola whose vertex is $(0, 0)$ and axis is

f x axis

so you get this and y equals to $4x^2$ is a parabola whose vertex is zero zero and axis is y axis therefore it will be drawn like this

so area

bounded between these parabolas elsewhere they will not intersect each other so area bounded between these parabolas will be now what we do we divide this area into

elementary area

so if you if we assume that width of one such elementary rectangle is dx then elementary area for this region will be $y \cdot dx$ this is your y is square equals to $2x$

and this is your y equals to $4x^2$

so $2x \cdot dx$ this is $y \cdot dx$ equals to $2x \cdot dx$

x

so this is actually $2x$ and this is $4x^2$

so elementary area will be

$2x \cdot dx - 4x^2 \cdot dx$

so to find out this integral we need

to find out the point of intersection of both parabolas

so to do that let

us compute the intersection of both

so if you substitute y here

you get $16x^4 = 2x$

so that will give you $8x^3$

$x^3 - x = 0$

so point of intersection by solving this

you will get $x = 0$ and

so you get $x^3 - 1 = 0$ you get $x^3 = 1$

$8x^3 - 1 = 0$ in one bracket into x so you get $x = 0$ and $x = \frac{1}{2}$

so one point of intersection is $(0, 0)$ another point of intersection is $(\frac{1}{2}, 1)$

you if you put $\frac{1}{2}$ here you get one so

so another point of this is $(\frac{1}{2}, 0)$

the point of intersection is $(\frac{1}{2}, 1)$

so the limits of integration will be from

minimum to maximum that is $x = 0$ to $x = \frac{1}{2}$ now let us integrate it

so you get $\int_0^{\frac{1}{2}} (2x - 4x^2) dx$

so we get $2 \cdot \frac{x^2}{2} - \frac{4x^3}{3}$ from 0 to $\frac{1}{2}$

so we get $1 - \frac{4}{3} \cdot \frac{1}{8}$ which is

equal to $\frac{1}{3} - \frac{1}{6}$ this is equal to $\frac{1}{6}$

so required area

is $\frac{1}{6}$ let us take another example find out area of the region which is outside parabola $y^2 = 4x$ and inside the circle $x^2 + y^2 = 8$

so this equation of the circle can be

written as $(x - 4)^2 + y^2 = 16$

sixteen

so draw x and y axis remember one thing that in solving all these problems you should be very good in plotting

so this is a circle with center

$(4, 0)$ and radius 4

so you get the circle and this parabola will intersect the circle

so we

need to find out that

so $x^2 + 4 = 8$ replacing y^2 by $4 - x$ here we get $8 - x$

so x^2 is $4 - x$ $x^2 - 4 + x = 0$

so x equals to zero x equals to four

so the parabola intersects the circle at $x = 0$ and $x = 4$

so parabola will be drawn like this

so you are supposed to find out the area

of the region which is outside the parabola and inside the circle so

your required area is this now since this circle is symmetrical about x axis

and parabola is also symmetrical

about x axis this required area is also symmetrical about x axis

so required area will be if say this area is A

so this area will also be a required area is twice of A now if you assume that this is the elementary

strip let us compute this area which is lying above x axis which is outside the parabola

inside the circle

so if this is a strip elementary area of dx length height will be value of y from circle minus

value of y from parabola

so your elementary area will be y from circle

so you need to solve

it for y

so you get $16 - x - 4x^2$ and you get $4 - x$ and $y = \sqrt{16 - x - 4x^2} - (4 - x)$

so elementary area will be because you

are using the curves above x axis so you will be using $16 - x - 4x^2 - (4 - x)^2$

square minus under root $16 - x - 4x^2$

so this is your elementary area and limit will be from

$x = 0$ to $x = 4$ let us compute it further

so substitute $x - 4 = t$ in this part of this integral you get the limit will at $x = 0$ limit

will be $t = -4$ and $x = 4$ $t = 0$ will be zero and you get here $16 - t^2 - 4t^2$ and this integral we can write as it is

so this will be equal to $\int_{-4}^0 \sqrt{16 - 5t^2} dt$

integral will be $\frac{1}{2} \left[t \sqrt{16 - 5t^2} + \frac{16}{\sqrt{5}} \sin^{-1} \frac{t\sqrt{5}}{4} \right]_{-4}^0$

by two $16 \sin^{-1} \frac{t\sqrt{5}}{4}$ minus 2 minus this limit from here is

minus 4 to zero and limit here will be from zero to 4

so you get 0 here then 0 then

at -4 again it is 0 then $-\frac{1}{2} \left[4 \sqrt{16 - 5 \cdot 16} + \frac{16}{\sqrt{5}} \sin^{-1} \frac{-4\sqrt{5}}{4} \right]$

minus 32 to the power $\frac{3}{2}$ minus

\sin is also plus zero

so finally you get the value of this is $-\frac{1}{2} \left[4 \sqrt{16 - 5 \cdot 16} + \frac{16}{\sqrt{5}} \sin^{-1} \frac{-4\sqrt{5}}{4} \right]$

π by two

so you get $4 \pi - 4 \sqrt{3}$ into 8

so final value is 4π

minus $32 \sqrt{3}$ is your final answer

so this is area which lies above x axis
 so required area will be twice of this
 so you get $8\pi - 64$ by
 3 let us take another example find out area bounded between circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$ let us draw both of them on cartesian plane
 so this first one $x^2 + y^2 = 4$ is a circle of cent radius two and center zero
 so you get this and this is point is $(2, 0)$ and this is also a circle with center $(2, 0)$ and radius two
 so you get this circle
 so the common area between these two circles is this to find out this let us divide this area into horizontal very thin horizontal rectangles like this and say this is dy width of one horizontal strip is dy
 so elementary area will be x goes from here to here
 so we need to find out equation of this portion of the circle we need to find out equation of this portion of the circle and this portion of the circle
 so equation of this portion and this portion
 so equation of this portion if you solve it $x^2 + y^2 = 4$ you get $x = \pm \sqrt{4 - y^2}$
 under root four minus y^2
 so equation of this will be $2 + \sqrt{4 - y^2}$ similarly you solve it solve this equation for x you get $x - 2 = \pm \sqrt{4 - y^2}$
 so $x = 2 \pm \sqrt{4 - y^2}$
 minus under root $4 - y^2$ now if you solve this for x you get two branches
 and
 so if you draw this vertical line $2 + \sqrt{4 - y^2}$ represents this portion of the curve i repeat $2 - \sqrt{4 - y^2}$ represents this portion green colored portion of the circle and $2 - \sqrt{4 - y^2}$ square will be represented by this red portion of the circle
 so the equation of this red portion is $2 - \sqrt{4 - y^2}$
 so therefore elementary area is this minus this into dy
 so that is $(2 + \sqrt{4 - y^2} - (2 - \sqrt{4 - y^2})) dy$ which is equal to $2\sqrt{4 - y^2} dy$
 so required elementary area if integrated you get the actual area that is integration of $2\sqrt{4 - y^2} dy$ from minimum to maximum again we need to solve an equation of both the circles to get this limit
 so to solve that if you put $y^2 = 4 - x^2$
 equals to $4 - x^2$ over here $(x-2)^2 + y^2 = 4$ we get the point of intersection
 so $x = 1$
 so at $x = 1$ $y = \pm \sqrt{3}$ put $x = 1$ here in either

of the equations say put x equals to 1 in this equation you get y as plus minus root three

so this is minus root three y equals to minus root three and this is y equals to plus root three

so the limit of the integration will be from minus root three to plus root three let us solve this integral this is you can see that this integral is from minus a to a $f(x) dx$ type where this integrand is even function

of y therefore you can write this as 0 to root 3 twice of 0 to root three two

four minus y square minus two dy so this is equal to four zero to root three that will be equal to for one by two well of the integral zero to root three under root four minus y square dy will be one by two a square is four sine inverse y by two minus in integral of this one is y zero to root three let us evaluate this and get the final answer

so we get this is equal to $4 \cdot \frac{1}{2} \cdot 2 \cdot \sqrt{3}$ this is $4 \cdot \sqrt{3}$ so you get $1 + 2 \sin^{-1} \frac{\sqrt{3}}{2}$ minus at zero it will be zero at zero it will be zero and this also will be zero

so we get this so for two this is $\frac{8\pi}{3} - 2\sqrt{3}$ three

so we get eight $\frac{\pi}{3}$ by three eight $\frac{\pi}{3}$ by three minus two root three this is the final answer now let

us take another simple example of a function which changes its sign and lies above and below x axis and see how the area can be computed let us take a simple example find out area bounded between x axis sine x and x equals to minus $\frac{\pi}{2}$ to x equals to three $\frac{\pi}{2}$

so let us draw the curves say this is minus $\frac{\pi}{2}$ this is $\frac{\pi}{2}$ this is $\frac{3\pi}{2}$

so $\sin x$ will exhibit approximately this kind of property which is the usual property of $\sin x$

so you get this curve so the required area is a one a two and a three summation of all three

so your required area let me draw let me shade it

so required area is this plus this is this since a_1 and a_3 are they are lying below x axis

so they'll be having negative value

so required area is equal to modulus of a_1 plus a_2 plus modulus of a_3

so a_1 is minus $\frac{\pi}{2}$ to zero $\sin x dx$ integration of $\sin x$ is minus $\cos x$ minus $\frac{\pi}{2}$ to zero minus one a two is zero to π $\sin x dx$ which

is minus $\cos x$ zero to π

so you get one minus minus one which is two a three is π to three $\frac{\pi}{2}$ by two $\sin x dx$ which is equal to minus $\cos x$ π to three $\frac{\pi}{2}$ by two which is equal to minus one

so required area will be is equal to for let us take another example find out area bounded by $\cos x$ plus $\cos y$ equals to one

so if you examine carefully this equation represents four curves $x + y = 1$, $-x + y = 1$, $x - y = 1$, and $-x - y = 1$.

$y = 1$ and $-x - y = 1$.

so let us plot them this is in the first quadrant $-x + y = 1$

so x is negative y is both

so this

is in this line $-x + y = 1$ this is $x + y = 1$ $x - y = 1$ $-x - y = 1$

$y = 1$ is this line x

$-y = 1 - x - 5 = 1$ is this

so the required area is therefore this let us divide this area into thin vertical

strips like this

so what you can see is for this portion which is on negative side of x axis the elementary rectangles start

from this line and finish on this line for this portion which is on the positive side of x axis they start from this line and finish on this line

so we need to break this total integration in two parts so

will be computing the area shaded by black and area shaded by red separately and then we will add it

so for black shaded area elementary area will be if you take dx as width of the strip

so dx will be $y = 1 - x$ minus $y = x - 1$ into dx the limit of x will be from here

to here

so by observation you can see that this will be one comma zero this will be minus

one comma zero and this is origin as usual so the limit of this integration will be zero to one

plus for red shaded region for this if you take dx as width of elementary strip

elementary rectangle elementary area whatever so limit will be from minus one to zero and

elementary area will be $y = 1 + x$ minus $y = -x - 1$ dx $f(x)$

minus $g(x)$ try to recall the formula we discussed earlier

so we get zero to one two one

minus $x dx$ plus minus one to zero to one plus x into dx this is equal to

$\frac{1}{2} [1 - x^2]_0^1 + \frac{1}{2} [1 + x^2]_{-1}^0$ by putting the limits we see that we get for

upper limit this value is zero then minus lower limit will be giving you $\frac{1}{2}$ into minus

$\frac{1}{2}$ plus upper limit again will be giving you value lower limit will be giving you zero

so we

get two into half again

so we get one plus one which is equal to two

so total required

area is two let us take another example

so example is find out area bounded between the curves $y^2 = 4a - x$ and $y = mx$ we need certain conditions on a and m so

that we can at least plot these curves properly

so we assume that a is zero and m is

also positive

so we will solve this problem in our next lecture with this i stop and later we will consider some more complicated problems and explore more about definite integrals thank you you

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