

welcome students

so till now we have seen the method of substitution for definite integrals and several properties of definite integrals by using these properties we can solve very complicated problem of different integrals in a much simpler way

so let us try some more examples and explore how we can solve complicated problems in

a much simpler way let us take examples example one

so if i ask you to compute this integral and you start integrating it by different techniques you may be in trouble but if use properties of

definite integral then it becomes very simple

so you may try to think in this way that

since you have limits from minus two to two

so if you have a integral of this kind

and you know that if the function is even then this becomes zero to twice of zero to a $\int_{-a}^a f(x) dx$ for even function and this becomes zero for odd function

so we must first try to find

out whether this integrand is odd or even

so if $f(x)$ is this

integrand

so let us check it whether it is odd or even

so we get here you will get plus sign because it

is even power and \cos of minus x is $\cos x$ therefore we are getting that f of minus x is minus $f(x)$

so integrand is odd function hence value of the integral say it is i will be zero you can see that by using the properties of definite integrals you

can very easily integrate and find out the value to be 0 of a very complicated problem let us take another example one approach to solve this problem is you can try

to rationalize it by multiplying the corresponding conjugate say $\sqrt{x} - \sqrt{1-x}$

and then try to solve it but i will not be using that approach i will try to see whether we can use

properties of definite integral

so say this is i

so by using the property that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$\int_0^a f(x) dx$ is same as $\int_0^a f(a-x) dx$

so i is equal to zero to a zero to one $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}} dx$ sorry $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}} dx$ root one minus x root one

one minus x and root one minus one minus x dx that

is zero to one under root one minus x under root one minus x plus under root x dx so

if say this is 1 and this is 2 if we add these two equations we get on the right hand left

hand side we get $2i$ and on the right we get $\int_0^1 dx$

so you can see the numerator and

denominator is same hence get cancelled and hence gets cancelled

so we get one and

hence i is one by two

so you can see that it seems that the problem is very complicated

but you can very easily solve it by the using the properties of definite integrals let us take another problem example

so one simple approach to solve this problem you replace $\sin^2 x$ by $1 - \cos 2x$ by two I am not going to use this approach instead I

will be using property of definite integrals

so let us see whether this function is even or odd

so $\sin^2(-x)$ is $\sin^2 x$ therefore this function is even hence you can write this integral as $\int_0^{\pi} f(x) dx$ by two twice of $\int_0^{\pi/2} f(x) dx$

now let us use another property says that $\int_0^a f(x) dx$ is same as $\int_0^a f(a-x) dx$

so you can write this by using this as $\int_0^{\pi/2} \sin^2 x dx$ by two $\int_0^{\pi/2} \sin^2(\pi/2 - x) dx$ this will give you twice of $\int_0^{\pi/2} \sin^2 x dx$

is $\cos^2 x$

so you get $\int_0^{\pi/2} \cos^2 x dx$ now if you this is I

so say this is one and this

is two

so again by adding one and two you get $2I$ is equal to twice of $\int_0^{\pi/2} (\sin^2 x + \cos^2 x) dx$ which is one we know that

so we get $2I = \pi$

so value of the integral is therefore

so we got $I = \pi/2$

so value of the integral is therefore $\pi/2$ so

you can see that another a very beautiful use of two properties of definite integrals which can be used to solve a very complicated problem

let us take another example and see how simple the integrations are if you are able to

use the formula for definite integrals correctly at the very first instance it seems a very complicated problem as usual but again by using the property that $\int_0^a f(x) dx$ is same as $\int_0^a f(a-x) dx$ we can write that $I = \int_0^{\pi/2} (4 + 3 \sin^2 x) dx$

by $\int_0^{\pi/2} (4 + 3 \cos^2 x) dx$

so I will be therefore equal to $\int_0^{\pi/2} (4 + 3 \sin^2 x) dx$

now again add one and two we get $2I$ on the lhs and on the right we get $\int_0^{\pi/2} (4 + 3 \sin^2 x + 4 + 3 \cos^2 x) dx$

you know that $\log m + \log n = \log mn$

so by using that you can immediately see that you get $4 + 3 \sin^2 x$ upon

$4 + 3 \cos^2 x$ into $4 + 3 \cos^2 x$ by $4 + 3 \sin^2 x$ which gets cancelled

so you get $2I = \int_0^{\pi/2} \log 1 dx$

so $2I = 0$

therefore $I = 0$ hence I is zero I hope you must be learning how to use these beautiful

formulas of definite integrals and evaluate complicated integrals let us take one

more example this is our final example then we will be moving ahead towards application

of definite integrals in finding out area between line and a curve so

let us take this x^e to the power x

so we can use in integration
 by parts to find out this definite integral
 so if we say this is our
 function first and this is our function second
 so by part integration we get first
 function into integration of second
 so we get x equals to zero to x equals to one minus
 zero to one differentiation of first will give you one and integration is
 e to the power x dx
 so we get one minus zero minus e to the power x integration of
 this is e power x is from zero to one
 so we get e minus zero minus e minus e to the power zero
 which is e minus e plus one
 so answer is one
 so this is not the end we shall take
 up later more complicated problems miscellaneous exercises will be doing
 where
 we will be using again all these properties and solve lot of complicated
 definite integrals
 for the time being let us start application of definite integrals in our very
 first lecture we have discussed
 several problems where we have discussed several cases area bounded between
 a line and a curve area bounded between two curves area bounded
 between three curves and
 so on
 so we shall take all these cases one by one
 from here onwards let us take area under simple curve case one
 so assume that this is your y axis this is x axis and this is some function of
 x which is always positive this is the line x equals to n
 this is the line x equals to b this is the line y equals to zero and this
 curve you know that it is y equals to $f(x)$
 so how to find out the area
 so what we have
 done you know that we divided it into several very thin rectangles right
 so we can take the width of say one
 rectangle as dx in this case and height of this rectangle will be y height of
 this rectangle will be y
 so the area of elementary strip or rectangle is y into dx y is the height
 and dx is the width now if you in
 so you have
 this elementary area dA dA is elementary area
 so if you integrate this dA from x
 equals to a to x equals to b it gives you the required area let me shade it
 so required
 area it will give you the required area which is bounded between four curves f
 x
 equals to a x goes to b and y equals to zero
 so this will give you the required
 area A
 so formula is $\int_a^b y dx$ now there are cases there are cases where
 this trick will not work for example if you have a curve like this where x is
 given
 in terms of y and the area is bounded between two horizontal lines say y
 equals to c to y equals to d then how will you evaluate the integral

so instead of dividing the area by vertical strips we divide the area by horizontal strips and we say that this width of this rectangle elementary rectangular elementary strip is dy and height of this strip will be x governed by this equation

so elementary area is $x dy$ which is our elementary area now if we integrate it from y equals to c to y equals to d we get the required area so the formula in this

case will be y equals to c to y equals to d $x dy$

so this was case two let us see case three where your function is all below x axis

so this is your $f(x)$ which is all negative from a to b this is the line x equals to a this is landline x equals to b and this is the required area

so again the formula by similar logic will be a to b $f(x) dx$ but since $f(x)$ is negative throughout value

of a will be negative therefore for required area you have to take modulus of the final value

so required area will be mod of a now let us take another case where function is not throughout negative or not throughout positive that means its it changes its sign then what will happen so

let us take case 4 where you have a function this is your y axis this is your x axis and

you have a function which is changing its sign say this is a this is b and this point of intersection of the function $f(x)$ is c with x axis

so you want to

find out area of the function which is bounded between x equals to a and x equals to b and x axis

so in this case total area a you will not be able to get by directly integrating it from a to b

so you have to integrate and get say area a one from a to c say this is a one and this area is a two and you get area

a two by integrating it from c to d

so total area required will be a one will be positive because $f(x)$ is negative from c to d sorry sorry c to b is c to b since

function is throughout negative from c to b

so a two will be negative

so total

area a will be a one plus mod of a two now let us use all these facts and solve some very simple problems at the beginning say

for example let us find out area of a circle $x^2 + y^2 = a^2$

so if this is your x axis sorry this is your y axis and this is your x axis and the circle is this

so you know that the circle is symmetric about both x and y axis

so the total area of the circle is also symmetric

so if we evaluate

this area then we can multiply it by four

so if this area is a

so total area total area is 4 times area of this circle which is lying

in the first quadrant

so 4 into a how to get a

so draw a vertical strip of $d \times$ length

whose height is y

so a will be $y d \times$ x goes from here to here

so the center of

the circle is $(0, 0)$ and this point will be $(a, 0)$

so x value will start from 0 and it

will go to a and y value will be computed from the equation of the circle

so you get y equals

to $\pm \sqrt{a^2 - x^2}$

so for each value of x you get two values of y so

positive value of y will give you the upper branch of the circle which lies above x axis and

negative value will give you the lower branch

so this is equal to $4 \int_0^a \sqrt{a^2 - x^2} dx$ now integration of this is known to

you

so you can directly put the value this is x

so x goes from zero to a now put

the values of upper and lower limits you get

so at a this will be 0 because of this

term and this term will give you $\frac{1}{2} a^2 \sin^{-1} 1$ which is $\frac{\pi}{2} a^2$ minus at 0 this is $\frac{1}{2} a^2 \sin^{-1} 0$

so at 0 it will

be 0 because of x

so you get zero here and then at zero $\sin^{-1} 0$ will be zero

so you get zero

so final answer is πa^2 now we have done this calculation by taking vertical strip the same thing can be done by taking horizontal strip also

so let us see how to do that for the circle

so let us draw the circle again and

let us see how to do it by using horizontal strip let us take this

horizontal strip whose width is t y and length is x governed by the equation of the circle

so x will be in this case $\pm \sqrt{a^2 - y^2}$ since we are using

this branch

so positive value of x will be taken negative value will be giving this branch of the circle

so total area required area of the circle is equal to $4 \int_0^a x dy$ now what are the limits of y

so this

point is $(0, 0)$ and this point is $(0, a)$

so y goes from zero and it goes to a

so y goes from zero to a and value of x will be taken positive because you are on positive side

so for this branch $\int_0^a \sqrt{a^2 - y^2} dy$

$\int_0^a x dy$ again by using the same formula $\frac{1}{2} y \sqrt{a^2 - y^2} + \frac{1}{2} a^2 \sin^{-1} \frac{y}{a}$ y goes from 0 to a

so again at a it

will be 0 and at a this will give you $\frac{1}{2} a^2 \sin^{-1} 1$ and at 0 it will be 0 and at zero

again it will be zero

so you get again πa square let us take one more example
and find out area of ellipse $x^2/a^2 + y^2/b^2 = 1$ where $a > b$

so this ellipse will look like something like this again since this ellipse is symmetric

with respect to both x and y axis

so we may compute only one fourth of the area
and then multiply it by 4 to get the total area

so total area is 4 times this area A and this area A if you use vertical strip
will again be given by $y dx$ where x will take values from here to here

so for this ellipse this is center $(0, 0)$ this is $(a, 0)$ this is $(0, b)$

is $(0, -b)$ and this is $(-a, 0)$ so for this region x minimum is 0 and maximum is a

so we get 0 to a y value get it from the equation of the ellipse and to get that you need to solve it

so you get $y = \pm b \sqrt{1 - x^2/a^2}$

so y is $\pm b \sqrt{1 - x^2/a^2}$

so for each value of x you are

getting two values of y but since you are using the portion of the ellipse which is lying above x axis this

so you will be taking positive sign

so we get $b \sqrt{1 - x^2/a^2}$

$\int_0^a b \sqrt{1 - x^2/a^2} dx$

so we get zero to a which is equal to $4 \int_0^a b \sqrt{1 - x^2/a^2} dx$

square plus $1 - x^2/a^2$ sine inverse x/a total area is therefore $4 \int_0^a b \sqrt{1 - x^2/a^2} dx$

$\int_0^a \sqrt{1 - x^2/a^2} dx = \frac{1}{2} \left[\frac{x}{a} \sqrt{1 - x^2/a^2} + \sin^{-1} \frac{x}{a} \right]_0^a$

so $4 \int_0^a b \sqrt{1 - x^2/a^2} dx = 4 \left[\frac{1}{2} \left(\frac{x}{a} \sqrt{1 - x^2/a^2} + \sin^{-1} \frac{x}{a} \right) \right]_0^a$

zero and this will give you $2 \left[\frac{x}{a} \sqrt{1 - x^2/a^2} + \sin^{-1} \frac{x}{a} \right]_0^a$

at zero it is zero and this is also zero

so we get $2 \left[\frac{a}{a} \sqrt{1 - a^2/a^2} + \sin^{-1} \frac{a}{a} \right] = 2 \left[0 + \frac{\pi}{2} \right] = \pi a b$ similarly this we have done via vertical

strip let us do it by using horizontal strip horizontal strip

so let us draw the ellipse again this time will be taking horizontal strip whose

width is dy and length is x this point is $(0, b)$ this point is $(0, -b)$ this is

$(0, 0)$ this is $(-a, 0)$ equation of the ellipse

now you have to solve for x

so $x^2/a^2 + y^2/b^2 = 1$

so $x = \pm a \sqrt{1 - y^2/b^2}$

so since we are taking this portion

of the ellipse

so x will be positive under root and the required area

is four times area of ellipse is four times this area which is y goes from 0 to b $\int_0^b 2a \sqrt{1 - y^2/b^2} dy$

so we get $4 \int_0^b a \sqrt{1 - y^2/b^2} dy = 4 \left[\frac{y}{b} \sqrt{1 - y^2/b^2} + \sin^{-1} \frac{y}{b} \right]_0^b$

so value of this integral is four a by b one by two y under root b square minus y

square plus one by two v square sine inverse y by b 0 to b

so you get 0 plus at at b it is zero then at at

b it is one by two b square pi by two minus zero then again it is zero

so you get pi a b with these two examples of area and circles by

using both vertical and horizontal strips you can see how to compute area of simple curves let us

complicate the situation and let us find out area bounded between a line and occur

so in this series let us take

example one the area bounded between y equals to one and y equals to x square

so let us plot both of them first

so y equals to one is a

horizontal line parallel to x axis and y equals to x square is parabola whose vertex is zero zero and axis

is y axis

so required area is this

so we can use horizontal strip over here

so also you can see that this y equals to x square

is symmetrical about y axis

so the total area required is equal to twice of area shaded by green say

so twice of area a

so i will be using horizontal

strip

so we get this is d y and this height of the strip is x

so we get x t y is the area of this

horizontal strip and then total area will be if you put the values of y from here

to here

so y goes from zero this is zero and it goes to one

so y equals to one now x is root y

so x the value of x is governed

by the equation of the parabola because the horizontal step

is ending at the parabola

so height of this strip is

governed by function value from parabola

so x is therefore

root y

so you get 0 to 1 root y d y

so 2 y to the power 3 by 2 by 3 by

2 0 to 1

so you get four by three now let us do it by vertical strips

so by using vertical strips or vertical elementary area

elementary elementary rectangles

so let me draw it again this is

your y axis this is your x axis this is y equals to one this is y equals to x square

so if you take vertical step what problems

will occur

so if you take vertical strip

so what will happen by giving by

integrating y dx from here to here or even double by multiplying it by 2 you

will not get the required area because y dx if you apply for this parabolic

region you get this area which is not the required area so how to use this vertical strip to get compute the area to see that let us draw the figure again so what we can do is we can so required area is is this area minus so required area is area shaded by red color and minus area shaded by green color so area shaded by red will be equal to $y dx$ x goes from you have to find out this value and this value for that you to solve y equals to one and y equals to x square that will give you x equals to plus minus one so this is x equals to one and this is x equals to minus one so so area of the rectangle will be from minus one to one y is this y is coming from the line y equals to $1 - y dx$ x again goes from minus 1 to 1 but this y is being computed from the parabola so you have required area is minus one to one one dx minus minus one to one x square dx so this is equal to two minus x cube by three minus one to one so two minus two by three which is again equal to four by three same as the previous calculation so to compute this area of ah rectang area this rectangle we have taken this strip and to compute this area of ah the region which is lying below parabola we have taken this rectangle vertical strips and hence we have to subtract two areas to get the actual area which is this by using method of vertical strips let us take another example and then example is the area bounded between y equals to x y square equals to two minus x and y equals to 0 which is in first quadrant so let us draw it y equals to x is this line and y square equals to two minus x is a parabola whose vertex is two comma zero so we have this kind of situation and the required area is this the line the area bounded between the line the parabola y equals to zero and which is lying in the first quadrant so to solve this integral we need to find out this is two comma zero this is zero zero we need to find out coordinate of these two points so we solve it and see that x is equal to so this is one comma zero and this is one comma one so the total area required is a is a one plus a two a one is this area a two is this area so a one is given by $y dx$ x goes from 0 to 1 plus $y dx$ and for a two this x limit will be from one to two this y is equal to x and this y is equal to under root two minus x so total area required is a one plus a two a one is zero to one $x dx$ plus one to two under root two minus $x dx$ this is

equal to one
by two x square zero to one plus two minus x three by two by three by two
minus one to
two
so we get one by two minus zero plus at two it will be zero
so zero
minus at one it will be one
so minus two by three
so total area is
one by two plus two by three
so seven by six today we have seen how to find out area of simple curves and we
have
computed area of circles ellipse and then we move towards little complicated
cases
where we have seen how to compute area of region bounded between a curve and a
line some more problems are left
in this category where area we are looking for area of region
bounded between a curve and a line and more complicated examples will
be taken in further classes thank you you