welcome students
 so in this lecture we will see an
application of derivatives to calculate limits of   certain functions
 so more specifically we will
learn what is known as the lopital rules for   finding the limit of functions
written
as ratio of two functions
 so let me state we will learn the lopital rules
 so this is used to calculate limits of the form limit x approaching c of ratio
f x by g x where c is in extended real number by this we mean that is c is
either a real number or plus or minus infinity
 so first let us look at a special case suppose f x and g x are continuously
differentiable differentiable functions in some interval i containing c also
assume that f of c is equal to g of c and both are zero then and let us assume
that g prime at c is non zero then we can write f x by g x as f x minus f of c
divided by g x minus g of c since f of c is zero and g of c is zero and this we
can write as f
x minus f of c divided by x minus c over g x minus g of c over x minus c
 so this all these are valid if x belongs to i and x not equal to c
 so now we have written this f x by g x as
the ratio of f x minus f c by x minus c   and g x minus g c by x minus c now
note
that what we know is that the limit of   f x minus f c by x minus c is
nothing but the derivative of f at c now limit x going to c of f x minus f c by
x minus
c this is equal to f prime c since we have   assumed that f is differentiable at
c this limit
exists and is equal to the derivative at c and the limit of x approaching c of g
x minus g
c divided by x minus c is equal to g prime c also we are assuming that the
limit   of this denominator which is g prime c this
is given to be non-zero
 so therefore limit x going to c of f x minus f c   by x minus c over g x minus
g c by x minus c
this is nothing but f prime c over g prime c but this ratio is nothing but f x
by g x
for x not equal to c
 so therefore limit of   x going to c of f x by g x is equal to f prime c
divided by g prime c but note that this is equal to
limit of x going to c of f prime x   divided by limit x going to c of g prime x
this is because f prime x and g prime x are assumed to be continuous assumed to
be continuous at x equal to c hence limit of x going to c of f x by g x is equal
to limit of x going to c of f prime x divided by g prime x
 so this above rule the above rule which is valid for more general cases is
known as the lopital rule
 so this loptal is the name of a french mathematician and this is pronounced as
lopital
 so here h is silent
 so now i will state the lopital
rule in more general situation suppose limit of x going to c f x is
equal to limit x going to c g x   which is equal to zero or plus or minus
infinity that is is f x by g x limit x going to c this is of the form zero by
zero or infinity by
infinity

so if we have this limit   in one of these indeterminate form zero by zero or plus minus infinity by infinity then   this we apply this l'hopital rule and the second assumption is also assume that limit x going to c of f prime x by g prime x this exists
 so suppose we somehow know that the limit of the ratio of the derivatives of these functions f   and g as x approaches c this limit exists and we have that g prime x is non-zero for all x in the interval i accept possibly at x equal to c
 so we assume that there is some interval in which   g prime is non-zero for all x in that interval
except may be at c then the conclusion is then the limit as x approaches c of f x by g x this exists and this limit is nothing but the limit of f prime x by g prime x
 so here it is important to note that only if we have this f x by g x is of 0 by 0 or infinity by infinity form then we can write this limit as   the limit of f prime x by g prime x provided this limit of the right hand side exists if we do not   have limit in zero by zero form we cannot apply this l'hopital rule
 so let us see some examples first example let me take limit x going to zero of sine x by x
 so here if we see as x goes to 0   sine x approaches 0 and x approaches 0 also
 so this is of 0 by 0 form now if we look at the limit of x going to zero of the derivative of sin x by d by d x of x then this is equal to limit x going to zero the derivative of sine x if we know is cosine x   and derivative of x is one
 so we get this is equal to limit of cos x by one and the limit   of cos x as x approaches zero is nothing but cos zero divided by one
 so this is equal to one
 so what we got here is that the limit of the derivative this exists and the denominator   if you see the derivative g prime x this is equal to 1 for all x
 so it is nonzero
 so by the loptal rule limit of x going to 0 of sin x by x this is equal to one which we have calculated directly note that here we use the fact that the derivative of sine x is cosine x in fact if you remember the way we calculated the derivative of sine x is cos x we use it we used   the fact that limit of sin x by x is equal to one
but suppose you know this fact by some other means   then we can also evaluate the limit of sine x by x equal to one using this l'hopital rule second example let us look at limit x going to 0 of e to the x minus one minus x divided by x square
 so again if i   look at this is f x is e to the x minus one minus x g x is x square 0 the numerator is 0 denominator is also 0 and both numerator and denominator are continuous functions of x

so the limit of numerator is zero   the limit of the denominator is zero
 so by
applying l'hopital rule this is equal to limit x going to zero of the
derivative
of numerator that will give e to the x   minus one divided by two x this
is by using l'hopital rules now if we see this limit now e to the x minus one
as x approaches zero is e to the zero
minus one which is zero
 so this is still in zero by zero form
 so we can try applying lopital rule again
so if we look at now again the derivative of   numerator and denominator we get
the
derivative of numerator is e to the x   the derivative of denominator is
two this is again by using n orbital now e to the x is a continuous function
 so this
limit is nothing but e to the zero divided by two   which is equal to one by two
 so now remark like we see here in this example
here we had to use l'hopital rule twice   to be able to evaluate the limit
 so we might have to apply lopital rules several times in order to calculate the
limit let me mention some complications which can occur suppose we try to
evaluate limit of   x going to infinity of e to the x plus e to the
minus x by e to the x minus e to the minus x
 so here we see that as x approaches
infinity e to the x this approaches infinity   e to the minus x approaches zero
 so we get
this is of the form infinity by infinity form
 so we might be tempted to
use lopital rule directly
 so if we use l'hopital rules this is equal
to limit x going to infinity of derivative   of e to the x gives e to the x
derivative of e
to the minus x gives minus of e to the minus x   divided by derivative of the
denominator
gives e to the x plus e to the minus x now if we see again at infinity the
numerator goes
to infinity denominator also goes to infinity so   it is still of infinity by
infinity form here
i will write in sort l h to say that we are   applying l'hopital's rule
 so again if i apply
l'hopital rule we get limit x going to infinity derivative will give e to the x
plus e to the
minus x by e to the x minus e to the minus x which is the original limit itself
 so here we see that by applying l'hopital rule   several times also we won't
be
able to calculate this limit
 so we wont be able to calculate the limit by applying lopital rule directly
however if we put e to the x is equal to y then as x approaches positive
infinity y approaches infinity and then the limit becomes e to the x plus e to
the minus x by
e to the x minus e to the minus x   this is nothing but y plus e to the minus
x will be one by y by y minus one by y and this can be written as y square plus
one by y square minus one
 so therefore limit x going to infinity of e to
the x plus e to the minus x by e to the x minus   e to the minus x is nothing

but limit of y going
to infinity of y square plus 1 by y square minus 1 which we know how to calculate we can divide
by the highest power y square numerator and   denominator which is equal to limit of y
going to infinity of 1 plus 1 by y square by   1 minus 1 by y square and then this becomes one
plus zero by one minus zero
 so the limit is one or we can use l'hopital rule we could use lopital rule to write limit of y going to infinity y square
plus one by y square minus one this is   of infinity by infinity form
 so by l'hopital
we can write this as limit y going to infinity   of the derivative of the numerator gives
2y divided by denominator again gives 2y   and we can cancel this 2 by 2y and we get this is
equal to 1.

 so this example shows that sometime   we have to do some substitution before applying lopital rule we can look at another   example where applying l'hopital rule directly
will not get anywhere
 so suppose i write   square root x plus 1 by square root x divided
by square root x minus 1 by square root x
 so this again is of infinity by infinity form and if we use l'hopital rule directly then
this will be equal to limit x going to infinity   derivative of square root x
gives 1 by 2 square root x plus   we have x to the minus half this is minus half
x to the minus three by two then again we have one by two root x plus half x to the minus three by two now as x goes to infinity
the numerator here goes to   zero and the denominator also both terms goes to zero
 so this is zero by zero form if we apply again lopital rule we get limit x going to infinity this is half
x to the minus half
 so we get minus one fourth   x to the minus three by two and then plus this
is three by four x to the minus five by two by   this will be minus one fourth
x to the minus three by two this becomes minus three by four x to the minus five by two this is again zero by zero form
so applying la lopital rules   makes this expression more and more complicated
becomes more and more complicated however we can write simply limit x going to infinity
square root x plus one by square root x   by square root x minus one by square root x   as we can simplify this expression and
write this as x plus one by x minus one and then it is easy to see that this limit is one
either by dividing numerator and denominator by x   or you can use l'hopital rule here
and this is limit x going to infinity   of the derivative will give one by one
 so this is equal to one
 so these two examples were to show that you
should not apply lopital rule blindly but try to   do some simplification before

you apply l'hopital
now we will see that this l'hopital rule can be used for other indeterminate
forms forms like zero times infinity or infinity minus infinity ah 1 to the
power infinity 0 to the power 0 etc by somehow changing into zero by zero
or infinity by infinity forms
 so for example lets first calculate what is limit of x going to infinity of x
square times e to the
minus x
 so here if we see as x goes to   infinity x square goes to infinity e to the
minus
x goes to zero
 so this is infinity times zero form which we have seen is an indeterminate
form
but here we have this is a product of two   functions and not ratio of two
functions
 so to
be able to apply l'hopital rule first we have   to convert it into ratio of two
functions
 so we
can write this as limit of x going to infinity   of x square divided by e to the
x now if
we see the numerator it goes to infinity   denominator also goes to infinity
so we get infinity by infinity form therefore we can apply lobster rule and
we get this as limit x going to infinity   of the derivative of x square
gives 2 x derivative of   e to the x is e to the x this is
still infinity by infinity form
 so we apply l'hopital rule once more   and this gives limit x going to
infinity of 2 divided by e to the x now as x goes to infinity this   is
numerator goes to 2 denominator goes
to infinity
 so this is equal to zero
 so more generally we can show that limit of x going to infinity of x to the n
times
e to the minus x this is equal to 0 for any positive integer n this is because
we write this as
x to the n divided by e to the x   and we keep on applying l'hopital rule
 so the
denominator is always e to the x when you take   derivative you keep getting e
to the x whereas
the numerator is x to the n
 so when we take   derivative of x to the n the exponent is reduced
by one
 so if we take derivative n times then we   get a constant in the numerator and
denominator is
still e to the x therefore this limit will be zero second example let us look at
limit of   x going to zero from the right of
the function x times natural log of x
 so here we are taking the right hand
limit because log x is defined here we are taking limit x going to zero plus
because log x is defined only for x greater than zero now if we
see what happens this x as x approaches 0 plus we have seen that what happens
to
log x this approaches negative infinity   recall that the graph of log x is like
this at 1 log x is 0 and for x less than 1 the value of log x is

negative and as you keep decreasing the value of x   log x keeps going to negative infinity
 so this
limit is of the form zero times minus infinity   we have to convert this into zero by zero or
infinity by infinity form
 so so let us write x log x this is equal to we can
write this as log x divided by x to the negative one now this is the numerator goes to negative infinity   denominator goes to positive infinity so
this is negative infinity by infinity form
 so by l'hopital rule this limit
x going to zero plus of x log x   is equal to limit x going to zero plus of log x by one by x and this if i use l'hopital rule this is
equal to limit x going to 0 plus of derivative of   log x gives 1 by x derivative of 1 by x is minus 1
by x square and if we simplify this 1 by x divided   by minus 1 by x square is nothing but minus x
so this is limit x going to 0 plus of minus x which is equal to 0.

 so the limit of x log x as x
approaches zero plus is equal to zero let us try to see example where we have the
limit of the form infinity minus infinity
 so let us try to calculate limit x
going to 0 of 1 by x minus 1 by sine x
 so as x goes to 0 one by x approaches
plus or minus infinity from right and left   and sine x as x approaches zero approaches
zero
 so this is infinity minus infinity form now here what we can do is we can take the common denominator and write
this as sine x minus x divided by x sine x now if we see as x approaches 0 the numerator is approaching 0 and as   x approaches 0 denominator is also approaching 0
 so we get 0 by 0 form
 so we can apply lopital rule and
write this as limit x going to 0   of the derivative of the numerator
gives cos x minus 1 divided by   derivative of the denominator we use product
rule and get this as sine x plus x cos x now what happens as x approaches 0 cos x
minus 1 goes to cos 0 minus 1
 so that is 0   and the denominator has sine x and x cos x so
this also approaches 0
 so we get 0 by 0 form
 so let us try to apply l'hopital rule
again if we take the derivative again   we get derivative of cos x is minus sine
x divided by derivative of the denominator   sine x derivative is cos x and x cos
x will give plus cos x minus x sine x now if we put x equal to zero sin zero is zero but
in the denominator we have cos zero plus cos zero this is let me write down this as minus
sign x divided by 2 cos x minus x sin x and now we get this is equal to 0

divided by two
 so this is equal to zero
 so we are able to calculate
this limit is equal to zero by   using l'hopital rule twice after
converting this into zero by zero form now similarly we can look at limit of x
going to 1 plus of x minus
1 times tan of pi by 2 x
 so as x approaches 1 from the right hand side   x minus 1 this goes to 0 and
then
we have 10 pi by 2 x
 so tan x it goes to infinity positive
infinity as a you go to pi by 2 from the left and from the
right this goes to negative infinity
 so here we are taking the limit
as x approaches 1 from the right
 so pi by 2 x approaches pi by 2
from the right
 so this is equal to 0 times minus infinity to be able to use lopital rule   we
should convert it into zero by
zero or infinity by infinity form
 so lets try to write this as x
minus one by tan is one by   cotangent
 so we can write
this as cot of pi by two x now we get zero by zero form
 so if we apply l'hopital rule this is
equal to limit x going to one plus of   derivative of x minus one gives one
derivative of
cotangent is minus of cosecant square pi by 2 x times derivative of pi by 2 x is
pi by
2
 so we get this and this is nothing but limit x going to 1 plus of minus 2 by pi
times sine square pi
by 2 x because 1 by cosecant is sine   and now as x goes to 1 from the positive
side pi
by 2 x goes to pi by two
 so this is equal to minus   two by pi times sine square pi by two sine pi by
two is one
 so this is equal to minus two by pi another type of limit is suppose we have 0
by 0 form
 so suppose we write   limit of x to the power x as x approaches zero
from the right
 so this is zero by zero form now here what we do is let f x equal to x to the x
then if we take log natural log of f x
is equal to x times natural log of x now what we know is that we have seen that
limit x going to zero plus of x log x this
is equal to 0 this we calculated by writing   this as log x by 1 by x and then
using
l'hopital rule this limit is 0
 so therefore limit of x going to zero plus of log of f x is equal to zero what
we have to
find is what is the limit of f x
 so now f x is nothing but e to the
power log f x
 so therefore   limit x going to zero plus of f x is nothing but

limit x going to 0 plus of e to the power log f x and because exponential is a
continuous function   this is equal to e to the power
limit x going to zero plus of log f x this is because e to the x is continuous
for continuous function the limit of f of x is same as f of the limit and now
we have already evaluated that this limit is zero
 so this is equal to e to the
zero which is equal to one
 so this limit is equal to one next i'll show you
that the assumption that limit of the ratio of   f prime x and g prime x exist
is necessary
so let me write this as a remark note that if limit x going to c f prime x by g
prime x does not exist we cannot conclude that the limit of f x by g x does not
exist
 so what we have said is that if the limit
exists then the limit f x by g x also exists   and they are the same but even if
the
limit of f prime x by g prime x does not   exist that does not mean that the
limit
of f x by g x does not exist for example take f x equal to x plus sin x and g x
equal to x then limit x going to positive infinity of f x   is equal to infinity
which is also the
limit of x going to infinity of g of x now what about f prime x if we look at f
prime x this
is equal to 1 plus cos x   g prime x is equal to 1
 so if we look at f prime
x by g prime x this is equal to one plus cos x
 so the limit as x going to infinity of this   is limit x going to infinity of
1 plus cos x which does not exist this is because the limit of cos x at
infinity does not exist cos x it keeps   oscillating between negative one and
one so
there is no limit as x approaches infinity however the limit of x going to
infinity of f x by g x   this is equal to limit x going
to infinity of x plus sin x   divided by x which is can be written as limit
x going to infinity of 1 plus sine x by x and now what happens to sine x by x as
x   approaches infinity we know that sine x
is bounded between negative one and one   denominator x goes to infinity
 so therefore this
sine x by x goes to zero
 so and sin sin x by x in mod is less than equal to one by x and
greater than equal to zero and one by x   this goes to zero as x approaches
infinity
so we have seen that by the sandwich theorem limit x going to infinity of sin x
by
x this is equal to 0 therefore limit of   x going to infinity of f x by g x is
equal
to one plus zero which is equal to one though if we directly try to use
l'hopital rule
then we get the limit of f prime x by g prime x   which does not exist but that
does not mean
that this original limit does not exist
 so with this i will stop this lecture thank you   you