

welcome to the next lecture on applications of derivatives

so in this lecture we will learn how to find the equation of the tangent line and the normal line at points on a curve and then we will see some application of that we will start with the equations of tangents and normals

so suppose we have a curve and if we look at this point say  $p$  whose coordinates are  $x$  comma  $y$  and we have this curve is  $y$  equal to some  $f$  of  $x$  now the tangent line to this curve is the straight line passing through this point  $x$  comma  $y$  whose slope is

so slope of the tangent line is the derivative  $\frac{dy}{dx}$  at that point at  $p$

so this we have seen that because if we look at any two points on the curve and join the line segment the line joining those two points and if we take the limit as the point  $q$  approaches  $p$  then we get the derivative the slope is nothing but the derivative at this point  $p$  and the normal line this is the tangent line and the normal line is by definition the line which passes through this point again and which is perpendicular to the tangent line

so this is the normal line

so what we know is that slope of the tangent line at some point say  $x$  naught  $y$  naught is the derivative  $\frac{dy}{dx}$  at this point  $x$  naught  $y$  naught which is also equal to  $f'$  at  $x$  naught if  $y$  is given as a function of  $x$  and therefore the slope of the normal at  $x$  naught  $y$  naught is minus one by this  $\frac{dy}{dx}$  at  $x$  naught  $y$  naught

so now to write the equation of the tangent and normal recall that equation of the line passing through some point  $x$  naught  $y$  naught and having slope  $m$  is given by  $y$  minus  $y$  naught is equal to the slope  $m$  times  $x$  minus  $x$  naught this is a well known formula for the equation of line in the point slope form

so therefore equation of tangent at  $x$  naught  $y$  naught let me call this point  $p$  is  $y$  minus  $y$  naught is equal to slope here is the derivative  $\frac{dy}{dx}$  at  $x$  naught  $y$  naught times  $x$  minus  $x$  naught and the equation of normal at the point  $x$  naught  $y$  naught is  $y$  minus  $y$  naught is equal to minus one by  $\frac{dy}{dx}$  at  $x$  naught  $y$  naught times  $x$  minus  $x$  naught if this  $\frac{dy}{dx}$  at  $x$  naught  $y$  naught this is non-zero now what can happen is if we look at say this circle lets look at circle whose equation is  $x$  square plus  $y$  square equal to one

so the problem is find the equation of the tangent line at a point  $x$  not  $y$  naught on the circle  $x$  square plus  $y$  square equal to one

so if we look at this say equation  $x$  square plus  $y$  square equal to one this implies  $2x$  plus  $2y \frac{dy}{dx}$  this is equal to  $0$  this implies the derivative  $\frac{dy}{dx}$  is equal to minus  $x$  over  $y$  if  $y$  is not equal to  $0$

so for every point except if  $y$  is  $0$  we get this these two points one comma zero and minus one comma zero if  $y$  is not equal to zero we have a point except for these two points

so therefore if  $y$  naught is not equal to zero the slope of tangent at  $x$  naught  $y$  naught is  $m$  is equal to minus  $x$  naught by  $y$  naught and therefore equation of the tangent at  $x$  naught  $y$  naught is  $y$  minus  $y$  naught is equal to the slope minus  $x$  naught  $y$  naught times  $x$  minus  $x$  naught which you can also simplify and write as  $y$  naught times  $y$  minus  $y$  naught plus  $x$  naught times  $x$  minus  $x$  naught equal to zero or which is same thing as writing  $x$  naught  $x$  plus  $y$  naught  $y$  is equal to  $x$  naught square plus  $y$  naught square but  $x$  naught square plus  $y$  naught square is equal to one

so we get  $x$  naught  $x$  plus  $y$  naught  $y$  is equal to one this we have derived if  $y$  naught is not equal to zero now clearly here you can see that if we have  $y$  naught equal to  $0$  we have these two points on the  $x$  axis here the tangent line is the equation is  $x$  equal to  $1$  and here the tangent line is  $x$  equal to minus  $1$ .

so here what happens is that the tangent line is vertical

so at the points  $1, 0$  and  $-1, 0$  the equations of the tangents are  $x = 1$  and  $x = -1$  respectively

so here what happens is that the slope of the tangent line is infinite

so if the slope of the tangent line is infinite at the point  $x, y$  then the equation of the tangent is  $x = x$  now we will look at some examples on finding the equations of tangent lines and normal lines second example find the point at which the tangent to the curve  $y = \sqrt{4x - 3}$  has slope  $\frac{2}{3}$

so if we look at  $y = \sqrt{4x - 3}$   $\frac{dy}{dx} = \frac{1}{\sqrt{4x - 3}}$   $\frac{dy}{dx} = \frac{2}{\sqrt{4x - 3}}$  that is  $\frac{2}{\sqrt{4x - 3}}$

so slope of tangent at  $x, y$  is  $m = \frac{2}{\sqrt{4x - 3}}$  we have to find the points where the slope is  $\frac{2}{3}$

so therefore we have to solve for  $\frac{2}{\sqrt{4x - 3}} = \frac{2}{3}$  this implies  $4x - 3 = 9$  and that means  $x = 3$

so  $x = 3$

so we get only one value of  $x$  satisfying this slope is  $\frac{2}{3}$  and when  $x = 3$   $y = \sqrt{4 \times 3 - 3} = \sqrt{9} = 3$

so this is  $y = 3$  hence the required point is  $3, 3$  next problem find points on the curve  $x^2 + 4y^2 = 9$  at which the tangents are first parallel to the  $x$  axis and next points where the tangents are parallel to the  $y$  axis

so this is in fact if you recognize this as an ellipse you can easily see that this is an ellipse and these points are  $(2, 0)$  and  $(-2, 0)$  and  $(0, 3)$  and  $(0, -3)$

so from the figure you can see that the points where the tangent will be parallel to  $x$  axis are these two points and the points where the tangents are parallel to  $y$  axis or these points let us try to find this using what we have learnt just now

so we are given  $x^2 + 4y^2 = 9$  this implies if we differentiate with respect to  $x$   $2x + 8y \frac{dy}{dx} = 0$   $\frac{dy}{dx} = -\frac{x}{4y}$   $\frac{dy}{dx} = -\frac{x}{4y}$  now if we want the tangent to be parallel to  $x$  axis then for the tangents to be parallel to  $x$  axis what should be the slope the slope must be zero because slope of the line  $x = \text{constant}$  sorry the slope of the line  $y = \text{constant}$  is zero

so if we equate the slope to be zero we get  $x = 0$

so therefore we get  $x = 0$  for the tangent to be parallel to the  $x$  axis putting  $x = 0$  in the equation of the curve we get

so  $x = 0$   $4y^2 = 9$   $y^2 = \frac{9}{4}$   $y = \pm \frac{3}{2}$

so  $y = \pm \frac{3}{2}$

so  $(0, \frac{3}{2})$  and  $(0, -\frac{3}{2})$  are the points where the tangents are parallel to  $x$  axis which we also observed by looking at the graph of this ellipse  $(3, 0)$  and  $(-3, 0)$  are the points where tangents are parallel to the  $y$  axis now first the second part if the tangent is parallel to the  $y$  axis the slope must be infinite

so if we look at this  $\frac{dy}{dx} = -\frac{x}{4y}$  we must have  $y = 0$

so therefore  $y = 0$  which will imply  $x^2 = 9$   $x = \pm 3$

so  $x = \pm 3$  therefore  $(3, 0)$  and  $(-3, 0)$

zero are the points where the tangents are parallel to the y axis this again we have seen from the picture that at minus two comma zero and two comma 0 the tangents are vertical lines okay

so the next problem you need to find the equation of the tangent to the curve  $y$  is equal to  $x$  minus 7 divided by  $x$  minus two times  $x$  minus three at the point where it cuts the  $x$  axis

so first we need to find the points where this curve intersects the  $x$  axis

so putting  $y$  equal to 0 we get  $x$  equal to 7

so the curve cuts the  $x$  axis at the point seven comma zero now we will find the slope of the tangent line for that we need  $\frac{dy}{dx}$

so  $\frac{dy}{dx}$  we can use the quotient rule

so we have the derivative of  $x$  minus 7 times the denominator  $x$  minus 2 times  $x$  minus 3 minus  $x$  minus 7 times  $\frac{d}{dx}$  of  $x$  minus two times  $x$  minus three divided by the denominator squared and this is equal to the derivative of  $x$  minus 7 is 1

so we get  $x$  minus 2 times  $x$  minus 3 minus  $x$  minus seven times  $\frac{d}{dx}$  of  $x$  minus two times  $x$  minus three this is nothing but this is  $x^2$  minus five  $x$  plus six

so the derivative is two  $x$  minus five divided by  $x$  minus 2 square times  $x$  minus 3 square now note that we have to find the slope at the point 7 comma 0.

if i put  $x$  equal to seven in this this second term here is zero

so we only get

so the slope of the tangent at the point seven comma zero is  $m$  is equal to  $\frac{dy}{dx}$  at seven comma zero which is equal to  $\frac{7 - 2 \times 7 - 3}{7 - 2^2}$  minus  $\frac{0}{7 - 3^2}$  and this can be cancelled

so this is one by five times four

so one by twenty this is the slope of the tangent and now we can easily write the equation therefore the equation of the tangent at 7 comma 0 is  $y - 0$  is equal to the slope  $\frac{1}{20}$  times  $x$  minus 7 or  $20y$  is equal to  $x$  minus 7.

ok

so the next problem will look at where the curve is given in parametric forms so here we will try to find for the normal find the slope of the normal to the curve  $x$  is equal to  $a \cos^3 \theta$   $y$  equal to  $a \sin^3 \theta$  at the point where  $\theta$  is equal to  $\frac{\pi}{4}$

so here the only thing is that we are not given  $y$  as a function of  $x$   $y$  and  $x$  are given in terms of the parameter  $\theta$

so to find  $\frac{dy}{dx}$  we can use the chain rule

so the derivative if we find  $\frac{dx}{d\theta}$  this is equal to  $3a \cos^2 \theta$  and then we get minus  $\sin \theta$  and  $\frac{dy}{d\theta}$  is equal to  $3a \sin^2 \theta \cos \theta$

so we have calculated the derivative of  $x$  and  $y$  with respect to  $\theta$  and this implies  $\frac{dy}{dx}$  is equal to  $\frac{dy}{d\theta} \frac{d\theta}{dx}$  this is equal to  $\frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta}$  we can cancel three  $a$  three  $a$  and then one  $\cos \theta \sin \theta$

so what we get is this is equal to minus of  $\tan \theta$

so  $\frac{dy}{dx}$  is minus  $\tan \theta$

so therefore the slope of tangent when  $\theta$  equal to  $\frac{\pi}{4}$  is  $m$  equal to minus  $\tan \frac{\pi}{4}$  which is equal to minus one we want the slope of the normal so therefore slope of the normal is equal to one because normal is perpendicular to the tangent

so the slope is 1 okay the next problem find a point on the curve  $y$  is equal to  $x$  minus two squared at which the tangent is parallel to the chord joining the

curve joining  $(2, 0)$  and  $(4, 4)$

so these two points are on this parabola  $y$  equal to  $x$  minus two square and we need to find the point where the tangent is parallel to this chord joining these two points

so first lets calculate what is the slope of the chord joining  $(2, 0)$  and  $(4, 4)$  is  $m$  equal to  $y_2$  minus  $y_1$  by  $x_2$  minus  $x_1$

so  $4$  minus  $0$  by  $4$  minus  $2$  which is equal to  $2$  this is the slope of the line joining these two point  $(2, 0)$  and  $(4, 4)$

so therefore because we want the tangent to be parallel to this the slope of the tangent is also equal to  $2$  now if we look at  $y$  equal to  $x$  minus two squared this implies  $\frac{dy}{dx}$  is equal to  $2$  times  $x$  minus two

so if we want the slope to be equal to  $2$  we get this equation which implies  $x$  minus two equal to one which implies  $x$  is equal to three

so we get  $x$  equal to three and putting  $x$  equal to three we get  $y$  is equal to three minus two square which is equal to one hence the point is  $(3, 1)$  where the slope is parallel to the chord joining these two points ok

so we have seen some problems say on finding the slope or equation of tangents and normal at some point on the curve next we will look at application of these two find approximations to the value of function at some point

so application of the tangent line to approximation

so let me explain what we want to do

so suppose we have some curve  $y$  equal to  $f$  of  $x$  and suppose we have a point here which is  $(x, y)$

so this is the point  $p$  which is  $(x, y)$  now let us look at another point on this curve where this  $x$  coordinate is some  $x$  plus  $\Delta x$  and let us call the  $y$  coordinate here is as  $y$  plus  $\Delta y$

so this is the point  $q$   $(x + \Delta x, y + \Delta y)$

so here we have  $y$  is equal to  $f$  of  $x$  and  $y + \Delta y$  is  $f$  at  $x + \Delta x$

so suppose it is easy to compute  $f$  at  $x$  but not

so easy to compute  $f$  at  $x + \Delta x$  what we want is we would like to approximate  $f$  of  $x + \Delta x$  by some value which is easy to compute

so what we do here is we approximate it by the tangent line

so let us look at the tangent line at this point  $(x, y)$  and then if we look at this point here this point the  $x$  coordinate is let me write this point as  $r$  its  $x$  coordinate is  $x + \Delta x$  but now the  $y$  coordinate here will be  $l$  of  $x + \Delta x$  where  $l$  of  $x$  is the equation of where  $y$  equal to  $l$  of  $x$  is the equation of the tangent at  $(x, y)$

so because we know how to calculate the equation of this tangent line we can calculate what is  $l$  of  $x + \Delta x$

so that will give me this value which is here

so instead of getting this  $y + \Delta y$  we will get this value which is  $l$  of  $x + \Delta x$  now suppose this  $\Delta x$  is small

so if we take the limit as  $\Delta x$  tends to  $0$  then this  $f$  of  $x + \Delta x$  minus  $f$  of  $x$  this divided by  $\Delta x$  if we take this limit we know approaches  $f'$  at  $x$  this is the definition of the derivative that this difference approaches the derivative at  $x$

so what we are doing is

so if  $\Delta x$  is small then the approximation  $l$  of  $x + \Delta x$  for  $f$  of  $x + \Delta x$  is not too bad

so what let us write the equation the the equation of the tangent at  $(x, y)$  is given by because i am using  $y$  here let me write this as capital  $y$  minus  $y$  equal to  $f'$  prime  $x$  is this slope of the tangent line times capital  $x$  minus  $x$  that is  $y$  is equal to  $y + f'$  prime  $x$  times  $x$  minus  $x$

so at  $x$  equal to  $x + \Delta x$  capital  $y$  is  $y + f'$  prime  $x$  times  $x$  minus  $x$

is  $\Delta x$

so hence the linear approximation is what we are doing is  $f(x) + \Delta x$  this is being approximated by  $y$  is nothing but  $f(x) + f'(x) \Delta x$  note that this is not exactly equal to we are approximating this so there is some error in calculating this

so now we will look at some example example one we want to approximate square root of 36.

6 of course you must have learnt method to calculate the square root of any number

so this can be calculated but we want to approximate this

so what we do is first you have to choose what is  $f(x)$

so we take  $f(x)$  equal to square root of  $x$  we want the value of this function at 36.

6 now if you see that if we take  $x$  equal to 36 then the square root of 36 is equal to 6.

so we take  $f(x)$  equal to this and if we take  $x$  equal to 36 and  $\Delta x$  is equal to 0.

6 then what we want is we want  $f(x) + \Delta x$

so what we know is that  $f(x) + \Delta x$  can be approximated by  $f(x) + f'(x) \Delta x$  now what is  $f(x)$  is the square root of 36 plus  $f'(x) \Delta x$  is  $\frac{1}{2} \sqrt{36} \Delta x$  is 0.

6

so this is equal to  $6 + \frac{1}{2} \cdot 0$  which is equal to 6.

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so what we are calculating this square root of thirty six point six is approximately equal to six point zero five though this is not exactly this there is some error involved here let us look at one more example

so note that in the previous one square root of 36.

6 there is a method to calculate square root now suppose i ask you to compute the value of cube root of 25 now this is not easy to calculate exactly

so again what we do is we take the function  $f(x)$  equal to cube root of  $x$  and then  $f'(x)$  will be  $\frac{1}{3} x^{-2/3}$  and then we take  $x$  to be equal to now we have to look for the value near 25 for which calculating cube root is easy

so the perfect cube which is near to 25 is 27

so we take  $x$  equal to 27 and we want  $x + \Delta x$  to be equal to 25

so that is we take  $\Delta x$  to be equal to minus 2 and then cube root of 25 is same as  $f(x) + \Delta x$  which we will approximate by  $f(x) + f'(x) \Delta x$  which is equal to cube root of 27 plus  $\frac{1}{3} \cdot \frac{1}{27} \cdot (-2)$  and then we take  $x$  to be equal to now we have to look for the value near 25 for which calculating cube root is easy

so cube root of 27 gives me 3 minus  $\frac{2}{27}$  and 27

so this will give me 9.

so three minus two by twenty seven which is equal to seventy nine by twenty seven

so cube root of twenty five is approximately equal to seventy nine by twenty seven we can also use this to approximate the rate of change of quantity

so let me do one more example the radius of a sphere is measured as nine centimeter with an error of point zero three centimeter find the approximate error in volume

so we have a sphere volume of the sphere is given by  $\frac{4}{3} \pi r^3$  what we are given is  $r$  is equal to nine centimeter  $\Delta r$  is point zero three centimeter if  $\Delta v$  is the error in volume this is equal to  $v$  at  $r$  plus  $\Delta r$

$r$  minus  $v$  at  $r$  and we have seen that this can be approximated by the derivative  $v'$  times  $\Delta r$

so what we do is we just calculate therefore approximate error in volume is equal to  $v'$  times  $\Delta r$  which is equal to  $v'$  is  $4\pi r^2$  times  $\Delta r$  and then you put  $r$  equal to 9

so this is  $4\pi$  times  $9^2$  times  $0.03$ .

$0.3$  this much centimeter cube

so this finishes our lecture on applications of derivatives thank you