

hello everyone

so welcome to the next lecture on derivatives in this lecture we will continue our discussion of finding points of minima and maxima of functions

so let us first recall what are local maxima and minima of a function f of x

so point c in the domain of f of x is called a point of local maxima if there exists some real number h positive such that f of c is greater than equal to f of x for all x in the open interval c minus h to c plus h that is f of c is the maximum value of f of x in some small enough interval containing the point c similarly c is called a point of local minima if there exists h positive such that f of c is the minimum value of f of x in the interval c minus h to c plus h

so for example if we look at a function say we have this function if you look at this point look at let us look at these four points we call these points c_1 , c_2 , c_3 , c_4 then if you look at this c_1 if i take this interval and if i restrict this function to this interval from c_1 minus h to c_1 plus h then this f of c_1 is the maximum value in this interval even though this is not the maximum value of the function for all x for example this at this point c_3 the value of the function is bigger

so this is a point of c_1 is a point of local maxima we look at c_2 at c_2 again if i take a small enough interval containing c_2 then you see that this f of c_2 is the minimum value in this interval

so c_2 is a point of local minima and at these points if you see the derivatives exist at f at c_1 and c_2 and here we have f' at c_1 is 0 f' at c_2 is also 0 now if we look at c_3 this point again if i take an interval c_3 minus h to c_3 plus h then the function it attains its maximum value at this point c_3

so this c_3 is again a point of local maximum and c_4 is a point of local minima what we know is that at a point of local minima if c is a point of local maxima or local minima then either f' at c is equal to 0 or f' at c does not exist remember that we have proved this in previous lectures that if we have a point of local maxima or local minima and if the derivative exists then the derivative must be equal to 0 there also we have that let me write this as this is called the first derivative test

so let $f(x)$ be a function defined on an open interval i and then we have if $f'(x)$ changes sign from positive to negative see what happens here in this example if we have as we move across this c_1 f' is positive here because the derivative the function is increasing and then the function is decreasing as we move to the right of c_1

so f' is negative here

so this changes sign from positive to negative then change the sign from positive to negative as we move across c then c is a point of local maxima similarly if $f'(x)$ changes sign from negative to positive as we move across c then c is a point of local minima as you can see at the local minima f' changes from $f' < 0$ to $f' > 0$ as we move across this point c_2

so this is a point of local minima and if $f'(x)$ does not change change sign as we move across c then c is neither a point of local maxima nor a point of local minima

so this gives us test to find points of local minima and maxima

so to find points of local minima and local maxima we find the critical points that is points where $f'(x)$ is equal to zero or $f'(x)$ does not exist and then we use we can use the first derivative test to determine whether those points are points of local maxima local minima or neither let us look at an example suppose we consider $f(x)$ equal to x^3 on \mathbb{R}

so if we look at $f'(x)$ is equal to $3x^2$

so what we get is $f'(x)$ is equal to $3x^2$ thus $f'(x)$ equal to

zero if and only if x is equal to zero that is zero is the only critical point now we will check whether 0 is a point of local minima local maxima or neither so if we see a this function since $f'(x)$ is equal to $3x^2$ this is positive for all x greater than zero

so if we see across this critical point 0 f' is positive f' is positive as we move from this

so f' does not change sign f' does not change sign as we move across x equal to zero this means that the function is increasing and the function is increasing here also

so in this case we see that this point x equal to 0 is neither a point of local maxima nor a point of local minima such a point which is a critical point but is neither a point of local maxima nor local minima is called an inflection point

so in this case x equal to zero is an inflection point or point of inflection

so let us look at one example find the points of local maxima and local minima of $f(x)$ which is given by $x^3 - 3x + 3$

so we find the derivative $f'(x)$ this is equal to $3x^2 - 3$ which is equal to $3(x^2 - 1)$ or $3(x - 1)(x + 1)$

so first we find the zeros of $f'(x)$

so $f'(x)$ equal to zero if and only if x is equal to -1 or x equal to 1 and then we see the sign of $f'(x)$

so we have critical points -1 and 1 and then we see that this $f'(x) = 3(x^2 - 1)$ is negative if x is between -1 and 1 and if x is greater than 1 then $f'(x)$ is positive $x^2 - 1$ is positive and also if x is less than -1 then $f'(x)$ equal to $x^2 - 1$ times 3 this is positive

so that means that the function is increasing it changes sign from positive to negative as we move across -1 and it changes sign from negative to positive as we move across the point x equal to 1 thus x equal to -1 is a point of local maxima and x equal to 1 is a point of local minima another example

let us look at $g(x) = 2x^3 - 6x^2 + 6x + 5$ here again let's find the derivative $g'(x) = 6x^2 - 12x + 6$ which is equal to $6(x^2 - 2x + 1)$ which we see is equal to $6(x - 1)^2$

so here again x equal to 1 is a critical point but we see that $g'(x)$ is positive as we move across x equal to 1 .

so here 1 is the critical point g' is positive to the left of 1 as well as to the right of 1 .

so the function is increasing and it is increasing in this interval

so in this case x equal to 1 is a point of inflection for $g(x)$ its neither a local maxima nor a local minima

so for this function $g(x)$ there is no $g(x)$ has no local max or local min we will later see that we can also draw the graph of this function $g(x)$ using these informations now let us look at some examples

so let us look at simply this function $f(x) = x^2$ if we find the critical point $f'(x)$ is equal to $2x$

so x equal to zero is the only critical point we can use the first derivative test to see that x equal to zero is a point of local minima because $f'(x)$ is negative it changes from negative to positive

so this is a point of local minima in fact in this case the function is simply you know the graph of this is this parabola and the function is always non negative and it is zero at zero

so it is clear that this point is a point of local minima it is also a point of global minima now if we look at what happens to the second derivative

so the first derivative does not tell us if we look at another function g of x for g of x equal to say minus x square x equal to 0 is a point of local maxima this is f of x equal to x square and if i take g of x equal to minus x square this is g of x minus x square here x equal to 0 is again a critical point which is local maxima

so for those of both of these f prime 0 is 0 g prime 0 is 0 .

let's look at the second derivative what is f double prime of x this is equal to 2 and if i look at g double prime of x this is equal to minus 2 .

so what we see is that

so in this example the second derivative of the function is positive at the point of local minima and for g of x this has local maxima at zero here the derivative is negative and is negative at the point of local maxima now the question is can we use the second derivative to test whether the function is local maxima or local minima at a point

so we'll discuss the second derivative test

so let me write as a theorem suppose f of x is a function which is twice differentiable on an open interval i also suppose f prime at c is equal to zero so we have a critical point at x equal to c now we have we want to decide whether c is a point of local maxima local minima or neither

so then first is if the second derivative f double prime c is greater than 0 then c is a point of local minima this we have seen in this example that f of x equal to x square the second derivative is positive at zero and this is the point of local minima second thing is that if the derivative second derivative is negative at c then c is a point of local maxima and third if the second derivative at c is equal to 0 then the test fails that is we cannot conclude anything if f double prime c is equal to zero

so let us see first that the third condition consider f of x equal to x to the four and g of x equal to minus x to the four then f prime 0 is 0 g prime 0 is also 0 also the second derivative at 0 is 0 second derivative of g is again zero and if we see directly we can see by the first derivative test or by direct observation that f of x has a local minima at x equal to 0 whereas g of x has a local maxima at x equal to zero

so what we see is that if the second derivative is zero at a critical point then it can be a local minima it can be a local maxima it can also be neither if we consider say h of x equal to x cube then we see that h prime x is three x square h double prime x is equal to six x

so in this case we again see that h prime 0 is 0 h double prime 0 is also 0 but we know that here x equal to 0 is neither a local max nor a local min

so by just looking at the second derivative at a critical point if it is zero then we cannot conclude anything

so if the second derivative f double prime c is equal to zero at a critical point c then we can have all possible cases in such case we may try to use the first derivative test now let us look at proof of the second derivative test

so the first case is suppose f prime c is 0 and the second derivative at c is less than 0 .

we want to prove that claim is that in this case c is a point of local minima so for this what we have to do is this we need to find some h positive such that f of c is less than equal to f of x for all x belonging to c minus h to c plus h now let's look at this information that f double prime c is negative and f prime c is zero

so by the definition of derivative what we have is that f double prime c this

can be written as the limit the second derivative is the derivative of first derivative

so this is the limit of $f'(x) - f'(c)$ divided by $x - c$ as x approaches c and this is the definition of the second derivative at c now what we know is that $f'(c)$ is zero but we have $f'(c)$ is equal to zero

so the limit as x going to c of $f'(x)$ divided by $x - c$ this is equal to $f''(c)$ and this is given to be negative $f''(c)$ is given to be negative sorry the the first case we are considering is suppose $f''(c)$ is positive

so this is positive then we want to show that this is point of local minima if the second derivative is positive then we will get a point of local minima

so we are given that this limit is positive what does that mean if the limit is positive then that means that if i take x to be we have this is the point c and we have some $c - h$ to $c + h$ this means that if i e take h small enough then the value of this must be positive

so this implies there exists h positive such that this function whose limit is positive this function $f'(x)$ by $x - c$ this must be positive for all x belonging to $c - h$ to $c + h$ what does that mean this means that if we look at if x is bigger than c and less than $c + h$ then this denominator $c - x$ is positive

so that will mean that $f'(x)$ must be positive this implies that if x belongs to c to $c + h$ if x belongs to c to $c + h$ then $f'(x)$ must be positive this is because the denominator is positive in this case and if x is less than c if it is in $c - h$ to c then $x - c$ is negative and we want this ratio to be positive then $f'(x)$ must be negative

so what we see is that $f'(x)$ the sign of $f'(x)$ it is negative for less than c and it is positive for greater than c that means that the function is decreasing and then increasing which means c is a point of local minima

so therefore by the first derivative test x equal to c is a point of local minima the second case is similar if $f'(c)$ is 0 and $f''(c)$ is negative then x equal to c is a point of local maxima can be proved in a similar way

so in this case what we will have is that we have this limit is equal to $f''(c)$ this is assumed to be less than zero if this is less than zero then we have for x belonging to c to $c + h$ $f'(x)$ must be negative and for x in $c - h$ to c $f'(x)$ must be positive that means that f' changes sign from positive to negative as we move across c

so then by the first derivative test it must be a point of local maxima now we will try to use this for a function find the points of local minima and local maxima of $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

so what we do first find the critical points

so we find $f'(x)$ is equal to $12x^3 + 12x^2 - 24x$ and $f''(x)$ is equal to $36x^2 + 24x - 24$ now first we find critical points for that we need to solve for $f'(x)$ equal to zero and $f'(x)$ is $12x^2 + 4x - 2$ equal to 0 which is same thing as $3x^2 + x - 1$ equal to zero

so x is equal to zero or one or minus two these are the critical points now we use the second derivative test

so we need to find the second derivative at this point

so now $f''(0)$ if i put x equal to 0 let me write down what is $f''(x)$ this is $36x^2 + 24x - 24$ so $f''(0)$ is -24

so $f''(0)$ is equal to -24 this is less than 0 this implies $x = 0$ is a point of local maxima if the second derivative at a critical point is negative then we have local maxima and f'' at other critical points are 1 and -2

so $f''(1)$ gives $36 + 24 - 24$ which is equal to 36 which is greater than 0 this implies $x = 1$ is a point of local minima and $f''(-2)$ is equal to

so here we can factor out 12 and then we have $3 \times (-2)^2 + 2 \times (-2) - 2$ which is equal to $12 \times 3 - 4 - 2$ which we see is positive

so this implies $x = -2$ is a point of local minima

so $x = 0$ is a point of local max and $x = 1$ and $x = -2$ are points of local min we can also use this to find what is the minimum value and maximum value

so then if we look at the value of the function at these points

so $f(0)$ this is equal to if you look at $f(x)$ is this

so $f(0)$ is equal to twelve

so this is the value at local max and $f(1)$ if you calculate this is $3 + 4 - 12 + 12$ this is equal to seven and $f(-2)$ if you calculate $f(-2)$ this turns out to be -20 ok

so then next one i will show you the example that we have considered before you consider $f(x) = x^3 + x^2 - 2x + 1$ find points of local minima and local maxima

so here like we calculated that first derivative $f'(x) = 3x^2 + 2x - 2$ and this gives $x = 1$ and $x = -1$ are critical points now if we calculate the second derivative $f''(x) = 6x + 2$ this is equal to $6x + 2$ if we evaluate at 1 this is equal to 8 which is positive this implies $x = 1$ is a point of local minima and if you calculate $f''(-1)$ this comes out to be -4 which is negative this implies $x = -1$ is a point of local maxima which agrees with what we have seen before that the graph of this function looks like this and at $x = 1$ we have a local minima this value is two and at $x = -1$ we have local at $x = 1$ we have a local minima and at $x = -1$ we have a local maxima

so with this i'll stop today in the next class we will see some more applications of derivatives thank you