

welcome all

so today we will continue our discussion on application of derivatives in particular today we will learn about finding the maximum and minimum values of function on a given interval if they exist and then we will learn about finding the tangent line equation of tangent line and normal lines

so let me first start with

so we will discuss about finding the maximum and minimum values of a given function f of x on a given interval say i

so first of all recall that i stated this theorem before that if f of x is a continuous function on a closed interval we say a b then the function attains its maximum value and minimum value on the interval a b which means that there exists points x naught and y naught in the interval a b such that f of x naught is less than equal to f of x is less than equal to f of y naught for all x belonging to this interval a b in this case f of x naught is the minimum value of f of x on the interval a b and it is attained at the point x naught which is in the closed interval a b similarly f of y naught is the maximum value of f of x on the interval a b and it is attained at the point y naught in a b note that this x naught and y naught need not be unique

so note that there can be more than one point at which the minimum or maximum value is attained for example we can have a function

so if you see this function attains this maximum value at these two points

so so these these two are points of maxima for this function in fact if we take f of x to be the constant function then the minimum as well as maximum value is that constant and it is attained at every point on the interval but there is only one but there is a unique minimum value and a unique maximum value now the question is how to find the minimum and maximum values

so let me give some examples example one suppose we have this function on an interval a b here if you see the maximum value of this function is this number capital m and the minimum value is this number small m and these are points

so this is my x naught and y naught

so in this example the minimum value and the maximum value can be attained at points in the open interval a b another case would be that these values suppose i have a function like this

so i take this function from minus one to one $f(x) = |x|$ on the interval $[-1, 1]$ in this case the minimum value is attained at x equal to zero which lies in the open interval $[-1, 1]$ whereas the maximum value is attained at x equal to plus minus one which are the end points of the closed interval $[-1, 1]$

so it is possible that the minimum value or maximum value can be attained at the end point also in this example at the point of minima the derivative $f'(x)$ does not exist whereas if you look at the previous example in this example the point at which the minimum and maximum are attained we see that $f'(x)$ is 0 and $f'(y)$ is also 0 .

so what we have seen from this example that the minimum value or maximum value can be attained either in the open interval a b or it might be attained at one of the end points

so conclusion the points of minima and maxima these let me call this as the points of absolute minima because this gives me the greatest value and the least value

so the point of absolute minima and maxima of f of x on a closed interval a b can be either one of the end points or an interior point another theorem that we have seen before recall theorem if f of x has minima or maxima on the open

interval a b then either either the derivative does not exist at that point for example in the example $f(x) = \sin x$ we have this or the derivative $f'(x)$ is equal to zero at that point

so such a point we will call critical points

so we write definition for a continuous function f of x on an interval a b the critical points of f are the points where the function where f is not differentiable or where the derivative equals zero

so now in order to find the the points of minimum and maximum what we have to look for

so to find the points of maxima and minima and the maximum and minimum values maximum value and minimum value what we do is the following step one find all critical points in the open interval a b that is the points where $f'(x)$ is equal to 0 or $f'(x)$ does not exist

so we check for the points where the derivative does not exist and then find the points where $f'(x)$ is equal to 0 this gives me all the critical points step 2 find the values of $f(x)$ at all critical points step 3 find the values of $f(x)$ at the end points that is find what is $f(a)$ and $f(b)$ and then we know that the minimum and maximum values are obtained at one of these

so find the minimum and the maximum values among those obtained in steps two and three this gives us the minimum and maximum value this gives us the minimum and maximum values and also the points of global minima and maxima that is the point where this minimum value and maximum value are obtained this global or this is also called absolute minima and maxima this is to differentiate from the concept of local minima and local maxima that we discussed before

so let us look at an example find the maximum and minimum values of $f(x) = x^3 - 6x^2 + 9x + 15$ on the interval let's say $[-1, 2]$

so what we do is first we find the critical points find the critical points

so here since $f(x)$ is a polynomial $f'(x)$ exists at all points

so there is no point where $f'(x)$ does not exist now if i find $f'(x)$ this is equal to $3x^2 - 12x + 9$ then to find the critical points we have to find the zeros of $f'(x)$ find the zeros of $f'(x)$

so we equate $3x^2 - 12x + 9 = 0$ that is $x^2 - 4x + 3 = 0$ and this gives us $(x - 1)(x - 3) = 0$

so $x = 1$ or $x = 3$ now $x = 1$ lies in the interval $[-1, 2]$ whereas $x = 3$ lies outside the interval

so in this case

so $x = 1$ is the only critical point in the open interval $[-1, 2]$ then we find the values find the values of $f(x)$ at the critical points critical points as well as the end points

so at the critical point $f(1)$ is equal to $f(x) = x^3 - 6x^2 + 9x + 15$ if you put $x = 1$ here we get $1 - 6 + 9 + 15$ this gives us 19 and then at the end points $f(-1)$ is equal to $(-1)^3 - 6(-1)^2 + 9(-1) + 15$ this is equal to $-1 - 6 - 9 + 15$

so this gives -1 and at the other end point $f(2)$ is equal to $2^3 - 6(2)^2 + 9(2) + 15$ and this is equal to $8 - 24 + 18 + 15$ this gives equal to 17 now among these

so we have these three values 19 $f(-1) = -1$ and 17 we look at the minimum and maximum among these

so the minimum value is minus one which is attained at at one of the end point minus one and the maximum value is 19 which is attained at the critical point x equal to 1 right

so this example shows how to find the minimum value and maximum value of a continuous function on a closed interval now consider another example suppose i define f of x equal to x plus 1 by x for x belonging to r minus 0.

so this function f of x is x plus 1 by x this is not defined at 0 but it is defined everywhere else now the question is find the minimum and maximum value of f of x if they exist

so first thing note that since this function f x equal to x plus 1 by x is defined for all real numbers except 0 we are not considering this function over a closed interval

so we do not know whether the minimum and maximum values are defined or not this minimum and maximum values exist or not

so if they exist we need to find that

so let me write this as remark since here we are not considering f of x over a closed interval the minimum value or the maximum value need not exist

so how to find this

so first of all note that this f of x equal to x plus 1 by x this can be written as x square plus one over x this is defined for all x not equal to zero now note that if i take x square plus 1 minus 2 x this is nothing but x minus 1 whole squared which has to be greater than or equal to 0.

so what does this say this implies that x square plus one must be bigger than equal to two x and therefore this implies x square plus 1 by x this is greater than equal to 2 if x is positive then i can divide by x and this will be greater than equal to 2 and if x is negative then this will become one by x of course in this case x square plus 1 is always positive and x is negative

so this will be less than 0 if x is negative

so if we consider

so on the interval if i take the interval instead of taking all non zero real numbers if i take the positive real numbers 0 to infinity f of x equal to x square plus 1 by x this is greater than equal to two

so this says that also at x equal to one if you see this if i put x equal to one then this is equal to zero and therefore at x equal to one f of x takes the value two hence f of x has minimum value to attained at x equal to one on the interval zero infinity now what about ah the maximum value is there any maximum value on the interval zero infinity

so if you see this function f x equal to x plus 1 by x note that this function this goes to positive infinity as x goes to 0 plus and also this goes to positive infinity as x goes to positive infinity

so therefore f x has no maximum value on the interval zero infinity now for the now on the interval if you look at minus infinity to zero we have f x equal to again x plus 1 by x

so if you see we are replacing x by negative x this is x and one by x these are negative

so this is same thing as negative of minus x plus 1 by minus x and now we know that here minus x is positive minus x is positive

so by using the previous one we can tell the

so therefore on minus infinity to 0 we have minus f of x this is equal to minus x plus 1 by minus x now here because minus x is positive we have already seen that this is greater than equal to 2 by previous part

so therefore f of x must be less than equal to minus 2 on minus infinity to

zero also as x goes to negative infinity $f(x)$ equal to x plus one by x this goes to negative infinity also and as x goes to zero from the left again $f(x)$ goes to negative infinity

so $f(x)$ has minimum value minus 2 obtained at x equal to at x equal to minus 1 if you evaluate you get minus 2 and it has sorry this is the $f(x)$ as maximum value minus two which is obtained at x equal to minus one and it has no minimum value on the interval minus infinity to zero another way is just like we did for the first part

so for x less than zero what we do is if i write $x^2 + 1 + 2x$ this is equal to $(x + 1)^2$

so this is again greater than equal to 0 which implies $x^2 + 1$ is greater than equal to $-2x$ and then this implies $x^2 + 1$ by if i write $-x$ here this will be greater than equal to 2 this is because here $-x$ is positive

so if i divide by $-x$ the inequality is same and this implies that $x^2 + 1$ by x this is less than equal to minus two which is same as what we have obtained here now can we use calculus to derive the same conclusion

so let us look at if we can do this

so we have $f(x)$ is given to be $x + 1/x$ for x not equal to zero now we know that this function $f(x)$ is continuous and differentiable for all non zero x lets calculate the derivative if we find $f'(x)$ derivative of x is one derivative of $1/x$ is $-1/x^2$

so derivative is $1 - 1/x^2$ now what we will do is we will see the sign of $f'(x)$

so if we see this $f'(x)$ this is equal to zero if and only if x^2 is equal to one that means x is equal to plus or minus one

so the derivative is zero at minus 1 and 1 also if we look at now the interval so we have minus 1 one

so if we look at this if x^2 is greater than one this implies $1/x^2$ will be less than one

so $f'(x)$ is $1 - 1/x^2$ this will be greater than 0 and if x^2 is less than 1 then $1/x^2$ is greater than one which implies $1 - 1/x^2$ is less than zero now x^2 is less than one means x is between minus one and one

so in this interval the derivative is negative

so $f'(x)$ is negative here and x^2 greater than one x^2 less than one is equivalent to x is between minus one and one and x^2 greater than one this is equivalent to x is greater than one or x is less than minus one

so the derivative is positive in the interval one to infinity and derivative is positive also in the interval minus infinity to minus one

so $f'(x)$ is positive on minus infinity to minus one and also on 1 to infinity and $f'(x)$ is negative on $(-1, 1)$ because at 0 the function is not defined

so on minus 1 to 0 $f'(x)$ is negative and also on zero to one $f'(x)$ is negative now recall that $f'(x)$ positive implies f is increasing in fact is strictly increasing in that interval and $f'(x)$ negative this implies f is decreasing

so what we know is that the function thus $f(x)$ is increasing on minus infinity to minus 1 and then $f'(x)$ from minus 1 to 0 is decreasing decreasing on minus one to zero and again it is decreasing on zero to one and increasing on the interval one to infinity

so if we see this what we have is we have minus one and one where the derivative is zero the function is increasing from minus infinity to minus 1 let us look at first for the positive part what happens

so on 0 to 1 the function is decreasing and increasing from 1 to infinity
so this implies let us draw this again minus 1 0 1 what we have is the function is increasing in this interval decreasing in this interval then again it is decreasing on this interval and increasing on this this means that this point at minus 1 we have this is a point of local maximum and one is point of local minima

so this first derivative test it tells you what happens to the function also what we know is that as x goes to 0 from the right the function goes to positive infinity

so we have at 1 the value of the function is 2 and then the function is decreasing up to 1 and then it is increasing from in 1 to infinity

so the graph of the function will look like this and at x equal to 1 the derivative is 0 and similarly on the negative side at x equal to minus 1 the value of the function is minus 2 and then the function is it goes to negative infinity as x goes to 0 from the left side and it also goes to negative infinity here and we have a local maximum at this point minus 1 right

so this is the how the graph of the function will look like

so from this we can conclude that on zero infinity $f(x)$ has no maximum value but has the minimum value two at x equal to one and on minus infinity to zero $f(x)$ as no minimum value but has a maximum value minus two at x equal to minus one note that if we consider over the whole interval on \mathbb{R} minus zero this function f of x equal to x plus one by x has no minimum value and no maximum value because the function tends to minus infinity as x goes to negative infinity and it tends to plus infinity as x goes to plus infinity

so it cannot have minimum or maximum value on the whole interval \mathbb{R} minus 0 on the whole less but if we take only on the positive real number then it has minimum value at two at x equal to one and on the negative real number it has maximum value minus two at x equal to minus one ok

so this finishes today's lecture in the next lecture we will learn about the second derivative test to find the a local minimum and local maximum and also some other applications of derivative thank you you