

welcome to the next lecture on derivatives

so in the last lecture we learned two important theorems which are the Rolle's theorem and the mean value theorem and then we saw one application of the mean value theorem by proving that if a function is differentiable in an open interval n is zero in the open interval then the function must be a constant today we will see some more applications like what happens if the derivative in an interval is strictly positive or strictly negative

so let me write this as theorem the theorem is suppose f from a to b is a differentiable function

so then if the derivative $f'(x)$ is greater than zero for all x in the open interval a, b then $f(x)$ is strictly increasing on the open interval a, b and if $f'(x)$ is negative for all x in the open interval then $f(x)$ is strictly decreasing on the whole interval a, b we have seen the converse that if we have an a differentiable function and if it is strictly increasing then the derivative is positive and if it is strictly decreasing then the derivative is negative here we are saying that the converse is also true

so proof is again using the mean value theorem

so suppose we take any x_1 and x_2 do then we have to show that $f(x_1)$ is strictly less than $f(x_2)$

so then f is continuous on the closed interval x_1, x_2 and is differentiable on the open interval x_1, x_2

so therefore by the mean value theorem there exists some c in the open interval x_1, x_2 such that we have $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ this is equal to $f'(c)$ but we have assumed that the derivative is strictly positive on the whole interval

so this we know is strictly positive and this implies $f(x_2) - f(x_1)$ is positive this is because here $x_2 - x_1$ is positive

so $f(x_2)$ is greater than $f(x_1)$ whenever x_2 is greater than x_1 the second part is similar

so now let us try to use this to prove some inequality

so problem prove that $\sin(x)$ is always less than equal to x for all x greater than equal to zero

so we want to prove that $\sin(x)$ is always less than equal to x

so we can let $f(x)$ to be the function $x - \sin(x)$ and then proving that $\sin(x)$ is less than equal to x is same thing as saying that we have to prove that $f(x)$ is greater than equal to zero

so let's see what is $f'(x)$

so then the derivative $f'(x)$ is equal to one minus derivative of $\sin(x)$ is $\cos(x)$ and we know that $\cos(x)$ is always less than equal to 1

so this is greater than equal to 0.

since $\cos(x)$ is always less than equal to one

so what we have is that the derivative is greater than equal to zero therefore $f(x)$ is an increasing function

so note that in the previous theorem if we assume that the derivative $f'(x)$ is greater than equal to 0 then instead of strictly increasing function we will get that $f(x)$ is an increasing function that means non decreasing function because we have this $f'(c)$ is greater than equal to 0 then we have $f(x_2)$ is greater than equal to $f(x_1)$.

so we will use that this $f(x)$ is an increasing function also if you put x equal to 0 $f(0)$ is 0 minus $\sin(0)$ is equal to 0

so therefore for x greater than equal to 0 we must have that $f(x)$ is greater than equal to $f(0)$ since f is increasing function but this is equal to 0 that is $x - \sin(x)$ is greater than equal to 0 that is $\sin(x)$ is less than equal

to x for all x less than $\pi/2$ for all x non-negative of course this is not true for x less than zero similarly let us prove another inequality involving tangent of x so so that x is less than $\tan x$ for all x in the open interval zero to $\pi/2$ by two

so again we do the same thing we put $f(x)$ equal to $\tan x - x$ and we have to show that in the open interval 0 to $\pi/2$ $f(x)$ is strictly positive

so we calculate the derivative then $f'(x)$ is equal to derivative of $\tan x$ gives me $\sec^2 x$ minus derivative of x is 1 and $\sec^2 x - 1$ is $\tan^2 x$ and we know that $\tan x$ is this is strictly greater than zero for all x in the open interval zero to $\pi/2$ because \tan is \tan of x is 0 only in the integer multiple of π

so in the open interval 0 to $\pi/2$ \tan of x is always positive

so $f'(x)$ is greater than zero therefore $f(x)$ is strictly increasing on the open interval zero to $\pi/2$ this implies that $f(x)$ must be greater than $f(0)$ and what is $f(0)$ is $\tan 0 - 0$ which is 0 for all x and 0 to $\pi/2$.

that is $\tan x$ is greater than x for all x and zero to $\pi/2$ we will do one more problem problem 3 is prove that $|\sin x - \sin y|$ this is less than equal to $|x - y|$ for all real number x, y

so to prove this inequality we note that if we take $f(x)$ equal to $\sin x$ this is continuous and differentiable everywhere

so we can apply the mean value theorem by the mean value theorem given any x less than y there exists some c in the open interval x to y such that $f'(c)$ is equal to $f(y) - f(x)$ divided by $y - x$ that is $\sin y - \sin x$ divided by $y - x$ is equal to the derivative $f'(c)$ but $f'(x)$ equal to $\cos x$

so $f'(c)$ is $\cos c$

so this implies that if we take modulus $|\sin x - \sin y|$ divided by $|x - y|$ this is equal to $|\cos c|$ for some c and we know that cosine theta in modulus absolute value is always less than equal to one and this implies that $|\sin x - \sin y|$ this is if x is not equal to y this is less than equal to $|x - y|$ and of course if x is equal to y then both left hand side and right hand side are zero

so this is true for all x, y

so before seeing some more about the application of rolls theorem or mean value theorem let me do one more important theorem about continuous functions

so this is known as the intermediate value theorem in short we will write it so here the assumption is suppose f from some closed interval a, b to \mathbb{R} be a continuous function and let y lie between $f(a)$ and $f(b)$

so what we have is there is a function continuous function and we have this is $f(a)$ this is $f(b)$ and suppose there is some y which lies between $f(a)$ and $f(b)$ then the conclusion is that there exists some x here in the interval a, b such that $f(x)$ equal to y then there exists at least one x in a, b such that $f(x)$ is equal to y

so here we can assume that this $f(a)$ is either less than $f(b)$ or $f(a)$ is greater than $f(b)$

so either we have this or we can have $f(a)$ is bigger than $f(b)$ and then if we take any y here then again we have some x such that $f(x)$ equal to y

so the theorem is intuitively clear if you see if we have any continuous function then it must take this is called intermediate value theorem that because it takes all the intermediate values between $f(a)$ and $f(b)$ if we have a continuous function but we will skip the formal proof you should note that continuity is required note that continuity is required for the intermediate value theorem because otherwise we can have if we allow a discontinuous function then now we can have a function like this this is my a

this is b now if you see I have $f(a)$ here $f(b)$ and if I take any y here then there is no x for which $f(x)$ is equal to y because either $f(x)$ is less in this interval or $f(x)$ is here right one corollary to this intermediate value theorem is that suppose f is a continuous function on a closed interval a b and assume that $f(a)$ and $f(b)$ are of opposite signs that is the product $f(a) \cdot f(b)$ is negative then there exist at least one x in the open interval a b such that $f(x)$ is equal to zero this is like if I have say $f(a)$ is negative and $f(b)$ is positive then what it says is that if I have a continuous function then it must cross this x axis at least once

so this is the point x here where $f(x) = 0$ and this clearly follows from the intermediate value theorem because zero lies between $f(a)$ and $f(b)$

so since zero lies between $f(a)$ and $f(b)$ by the intermediate value theorem there exists x in a b such that $f(x)$ is equal to zero of course this x cannot be a or b because $f(a)$ and $f(b)$ are zero

so there is at least one x and you can have more than one x as well

so this function can be like this

so this intermediate value theorem is again very important

so some applications of intermediate value theorem

so the first one let me write this again as a theorem

so this theorem says that every polynomial of odd degree must have at least one zero note that one that is if $p(x)$ is equal to $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ to the n where n is odd and a_n is not equal to zero then there exist at least one c real numbers such that $p(c) = 0$ remark the result is not true for even degree polynomials for example if I take $p(x) = x^2 + 1$ this has no real zeros we know that $x^2 + 1$ is always greater than equal to one for all real x

so this cannot have a zero in \mathbb{R} but for odd degree polynomial we claim that there is at least one zero

so the proof of the theorem

so we have $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ since n is an odd integer if we look at the limit of x to the n the highest term this as x goes to positive infinity this is equal to positive infinity and the limit as x approaches negative infinity this will give us negative infinity this is because n is odd if n was even then both these will be positive infinity

so therefore if we look at this polynomial therefore $p(x)$ must be negative at some x and positive at some other x right this will depend on the sign of this a_n if a_n is positive then $p(x)$ will approach positive infinity as x goes to infinity and negative infinity as x goes to negative infinity that means that if you take x large enough then $p(x)$ must be positive and if x is large negative number then $p(x)$ must be negative and if a_n is negative then you have the other way

so then by the intermediate value theorem by the corollary to intermediate value theorem $p(x)$ must be zero at some point x

so this is a very important result that any polynomial of odd degree must have at least one real θ that means that this must cross the x axis at some point

so now let us look at one problem

so that $x^5 + 4x - 1 = 0$ has exactly one solution in \mathbb{R}
so first of all if we see that let $p(x)$ be this polynomial $x^5 + 4x - 1$ since $p(x)$ is an odd degree polynomial $p(x) = 0$ has at least one solution this we know because this is odd degree it has at least one solution what we want to show is that there is exactly one solution that means that we cannot have other solution to this

so suppose there are two solutions x_1 and x_2 then what we have is $p(x_1) = 0$ and $p(x_2) = 0$

so by the rolls theorem note that because we have polynomial this is continuous and differentiable everywhere and at the end point x_1 x_2 these values are same

so by the rolls theorem there exists some c in between x_1 and x_2 such that such that $p'(c)$ is equal to zero but if we look at $p'(x)$ the derivative here is $5x^4 + 4$ and we know that x^4 is always non negative this is always greater than equal to four

so therefore we get a contradiction

so the contradiction is because we assume that there are two solutions x_1 x_2 therefore $p(x) = 0$ has exactly one solution you might ask question that where is the solution what is the solution

so note that here we have a polynomial of degree five

so there is no general method to find the roots of this polynomial but what we can do is that we can try to we can use the intermediate value theorem to find approximate solution as follows what we have is we have $p(x)$ is equal to $x^5 + 4x - 1$

so note that if i put x equal to 0 $p(0)$ is equal to minus 1 this is negative if i put x equal to 1 then i get this is equal to 4 which is positive

so by the intermediate value theorem we know that $p(c) = 0$ for some c between 0 and 1 right

so you have to look for the root of this polynomial only in the interval 0 to 1 there is no other root

so there is no 0 outside this interval now what you can do is that if we look at the midpoint the value at the midpoint which is half

so if you see $p(\frac{1}{2})$ this is equal to one by thirty two plus four times half is two minus one now this is again this is $1 + \frac{1}{32}$ this is still greater than 0

so now you know that $p(0)$ is negative $p(\frac{1}{2})$ is positive

so there must be since $p(0)$ is negative $p(\frac{1}{2})$ is positive this zero must lie in the interval zero to half again you can repeat this process again find what is $p(\frac{1}{4})$ $p(\frac{1}{4})$ will be one by four to the five plus one minus one

so this is one by four to the five again it is positive

so the zero must lie in zero to one fourth then you can find at one eighth what happens and if at one eighth $p(\frac{1}{8})$ this gives one by eight to the fifth plus half minus one and this is equal to one by eight to the fifth minus half which is of course negative

so therefore the 0 must lie in the interval $\frac{1}{8}$ to $\frac{1}{4}$ previously we had that the 0 is between 0 to $\frac{1}{4}$.

now we are divided into sub interval 0 to $\frac{1}{8}$ and $\frac{1}{8}$ to $\frac{1}{4}$ and we know that the 0 must lie in $\frac{1}{8}$ to $\frac{1}{4}$ again you can take at the midpoint of this interval and this way you will get a better and better approximation

so proceeding this way we can get the zero lying in smaller and smaller intervals this method that we used here is called the bisection method because we sub divide the interval into half and half and then we see in which interval the zero lies

so this gives one method to find the where the zero of f function lie let us do some more problems

so prove that $\cos(x)$ is equal to x for some x in the open interval 0 to $\frac{\pi}{2}$.

so note that there is no general way of solving this equation $\cos(x) = x$ but still we want to prove that this has some solution in this interval zero to $\frac{\pi}{2}$

so to do such problems we use the intermediate value theorem

so you write let $f(x)$ be equal to $\cos(x) - x$ and then because we have to

prove that the zero of this function lie in this interval you find what is the value of this function at the end point

so $f(x)$ is continuous everywhere also if i find what is $f(0)$ this is cosine zero minus zero which is equal to one

so this is greater than zero what about f at $\pi/2$ this gives me cosine $\pi/2$ minus $\pi/2$ which is equal to one minus sorry this is equal to zero minus $\pi/2$

so this gives me a negative quantity

so $f(0)$ is positive $f(\pi/2)$ is negative this means that by the intermediate value theorem there exists some x in the interval 0 to $\pi/2$ such that $f(x)$ is equal to 0 .

that is $\cos x$ is equal to x right and again like we did for the previous one you can use the bisection method and then you find the value at $\pi/4$ you will get $\cos \pi/4$ minus $\pi/4$ you have to see whether that is negative or positive to determine in the interval of half of this length where the solution lies and you can keep doing like that to find smaller smaller interval where the solution lies let's look at one more interesting problem

so we assume that f is a function defined on closed interval 0 to r and this is assumed to be a continuous function also it is given that $f(0)$ is equal to $f(2)$ the value at the end points are equal what we have to show that prove that there exist two points x and y in this closed interval zero two such that $y - x$ is equal to one and $f(x)$ is equal to $f(y)$

so what we are given is we are given any continuous function from the interval zero to two and only thing we know is that $f(0)$ and $f(2)$ are same the value is same and then we have to show that this can be any function like this we have to show that there are two points where the value $f(x)$ and $f(y)$ are same and also that this x and y differ by one

so to solve this what we do is say we have to show that to find x and y such that y is equal to $x + 1$ and $f(x)$ is equal to $f(y)$ that is we need to

so the existence of x in the interval zero one such that $f(x)$ is equal to $f(x + 1)$ because y is $x + 1$ and we want this to lie in the interval 0 to 2 so therefore x must lie in the interval 0 to 1 .

so we look for a point in $0, 1$ where the value of the function is equal to $f(x + 1)$.

so this suggests that what we do is that we let $g(x)$ be equal to $f(x + 1) - f(x)$ and this function is defined for x belonging to the closed interval zero one

so if x is between zero and one $x + 1$ is between one and two

so $g(x)$ is defined in the interval $0, 1$ and this is continuous then g is continuous on the interval $0, 1$ also what is the value at the end point what we have to show is that g is 0 at some point

so we find the value at the end point $g(0)$ will give me $f(1) - f(0)$ and $g(1)$ is equal to $f(2) - f(1)$ what is given is that $f(0)$ is equal to $f(2)$

so this $f(2)$ is $f(0) - f(1)$

so $g(0)$ is $f(1) - f(0)$ $g(1)$ is $f(0) - f(1)$

so $g(1)$ is nothing but minus of $g(0)$

so thus by the intermediate value theorem there exists some x in the closed interval $0, 1$ such that $g(x)$ is equal to zero right these are of opposite signs

so there is some x such that $g(x)$ is zero and that says that that is $f(x + 1) - f(x)$ is equal to zero that is $f(x + 1)$ is equal to $f(x)$ hence we are done

so this problem again was application of intermediate value theorem but we had to define this new function g of x and then apply the intermediate value theorem so we will stop here in the next class we will learn some more topics on derivatives thank you you

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