welcome to the next lecture on derivatives
so in the last lecture we started with discussing two important theorems which are the rolls theorem and the mean value theorem
so let me recall what the rolls theorem says
so let me state the rolls theorem
so the assumption is that let f be a function from a closed interval a b to r set of real numbers b a function satisfying the following first is that f is continuous on the closed interval a b second assumption is that f is differentiable on the open interval a b and the third assumption is that the value of the function at the end point that is f of a is equal to f of b and then the conclusion is then there exists at least one c in the open interval a b such that the derivative f prime at c is equal to zero
so let us understand this by the picture we draw the graph of the function y equal to f of x what is given is that the value at a and b the function values are same
so this is f of a and f of b and then it is given that the function is continuous on the closed interval and differentiable on the in the open interval a b then we claim that the derivative
so the function might be like this here if you see that there is this point where we have horizontal tangent that means the derivative is zero or it can be something it can go down like this and go up in this case if you see there are two points where the derivative is zero
so there can be more than two points as well
so it can be it goes up and down like this and then you see that there all these points where the derivative is zero
so what we were trying to before explaining the proof of this theorem let us try to see that the assumptions in the theorem are necessary
so the first one says that f has to be f is assumed to be continuous on the closed interval a b suppose f is not continuous on the closed interval a b then you might have that the function might be just increasing like this and then i can define at the end point suppose this is a and this is b then at a i can define this value
so this function there is no c in the open interval a b such that f prime c is equal to 0 and the function here f is differentiable on a b f is f of a is equal to f of b f is continuous at all points of a b except at a
so even if it fails to be continuous at one of the end point then there need not be any c where f prime c is equal to 0.

so we required continuity in the closed interval a b the second assumption that f is differentiable differentiability of f on the open interval a b
so suppose this fails then we can have the function might be like this
so i have a b here and if you see this function again there is no point
so this this function is f is continuous on a b f is differentiable at all except one point in the open interval a b and f of a is equal to f of b and you can see that here for all point let us say this this is our point for any point to the left of this the derivative is constant positive one and for any point greater than this the derivative is negative one but there is no point where at this point the derivative is not defined
so there is no c such that f prime c is equal to zero and the third condition f of a equal to f of b of course this also we need because if i have say just this function
so here you see that the function is continuous on the closed interval it is differentiable in the open interval and
so if i write f of x equal to say x on the closed interval zero one then this

is the f is continuous on zero one f is differentiable on the open interval zero one and if you look at f prime of x this is equal to one for all x in the interval zero one

so there is no c where r f prime c is equal to zero but here we do not have here f of zero is not equal to f of one

so these three examples shows that all the three assumptions in the rolls theorem are necessary for the conclusion that f prime c is equal to zero at some point in the open interval a b if even one of them fails then the conclusion need not be true

so now i will give some idea about why this theorem is true

so idea of the proof

so if you note if you

so see this pictures then you can see that these points where the derivative is equal to 0 these corresponds to points of minimum or maximum of the function under the assumption of the rolls theorem we can show that there is at least one point c in a b where f of x attains its minimum or maximum value now one fact is that if f is assumed to be a continuous function on closed interval then it must attain its minimum and maximum value

so let me write this fact any continuous function f from a closed interval a b to r is bounded and f attains its minimum and maximum value on the close interval a b that is there exists point x naught y naught in the closed interval a b such that f of x naught is the minimum value that means f of x naught is less than equal to f of x and f of y naught is the maximum value for all x belonging to the closed interval a b note that the previous result is not true for continuous functions on open interval a b for example if you take say f x equal to tan x on the open interval minus pi by two to pi by two is continuous but there is no minimum there is no minimum value nor maximum value right this function tan x you must have seen that the graph of tan x between minus pi by two to pi by two looks like this

so as x goes to minus pi by two this goes to negative infinity as x goes to pi by two it goes to positive infinity

so this function is not even bounded but if we are have a continuous function on the closed interval then this must be bounded and it must attain the minimum and maximum value another fact is if f attains its minimum or maximum value in the open interval that is not at the end points then the derivative at that point must be zero if the function is if the function f is differentiable at that point

so here we have a function f this is defined on the closed interval a b to r and suppose the minimum or maximum value is attained in the open interval then there are two cases either f is not differentiable at that point or if it is differentiable then the derivative must be zero

so example here if i see this function from 0 to 1 and this is half the maximum value is attained at x equal to half but they function is not differentiable at x equal to half whereas if i have with the function to be differentiable if we have something like this then here again the maximum value is at half here if you see f prime of half is equal to zero

so we will use these facts to prove the rolls theorem

so first of all since f is assumed to be continuous on the closed interval a b there exists points x naught y naught in the closed interval a b such that f is minimum at x naught and it attains maximum at y naught this is true for all x in a b

so this is because any continuous function on a closed interval must attain its minimum and maximum value on that interval

so now there are two cases case one x naught and y naught are the end points a

and b

so in this case but since f of a is equal to f of b we must have that f of x is a constant on a b right because one of these f of a is minimum value as well as maximum value

so therefore f of x must be equal to this value for all x in a b and if f of x is constant we know that the derivative of constant function is 0 this implies f prime x is equal to 0 for all x in the open interval a b

so we can choose any c in a b to get f prime c is equal to zero

so in this case we just have a constant function f of a and f of b are equal and these are minimum as well as maximum value

so in this case f prime is zero at all the points in the open interval case two at least one of x naught or y naught lies in the open interval a b in this case what we have is that either the minimum if it is x naught in the open interval either the minimum or the maximum value of f x is attained in the open interval interval a b and we have stated this fact that if f attains its minimum or maximum value in the open interval and if the function is differentiable there then it must be zero

so then f prime must be zero at that point because f is assumed to be differentiable on the open interval a b

so this proves the rolls theorem ok let me explain why f prime must be zero at the point of maximum or minimum in an open interval

so what we have is suppose we have this point where we have the maximum value of the function is attained at this and

so suppose f of c is greater than equal to f of x for all x in a b where c is in the open interval a b also assume that f is differentiable at at x equal to c if it is not differentiable then we might have that the maximum is attained there and f prime c does not exist

so here we are assuming that it is differentiable here then we want to say that

so claim is f prime of c must be equal to zero suppose not then either f prime c is greater than zero or f prime of c is less than zero now what happens if f prime c is greater than zero if f prime c is greater than zero then this limit of f of x f of c plus h minus f of c divided by h this is equal to f prime c which we are assuming is greater than zero now if this is greater than zero then if this limit is greater than zero then for a small h this f of c plus h minus f of c by h this must be greater than zero because if it was less than equal to zero for all small n of h then in the limit this has to be less than equal to zero

so therefore we have this this implies that f of c plus h is greater than f of c for all small as contradicts f of c is the maximum value right

so the derivative greater than 0 will imply that this function must be the value must be greater than the value at f of c for x greater than c similarly if the derivative is less than 0 then that means that the function at this point it must go down like this

so similarly we get a contradiction if f prime c is less than zero hence f prime of c must be equal to zero

so this proves the rolls theorem where we use one fact that any continuous function on closed interval must attain its maximum and minimum value on it without proving it we assume that but we proved the rolls theorem using this and then using this fact that the derivative must be zero if the minimum or maximum value is attained in the open interval

so next we will prove what is called the mean value theorem

so the mean value theorem is a generalization of the rolls theorem here we assume that let f be defined from a closed interval a b to r b such that first f x is continuous on the closed interval a b and second f x is differentiable on

the open interval a b

so these two conditions are same as the first two conditions in the rolls theorem in rolls theorem we had third condition that the value of the function at the end points are equal f of a equal to f of b in the mean value theorem we do not assume that f of a equal to f of b then the conclusion is then there exists at least one point c in the open interval a b such that f prime of c is equal to f of b minus f of a divided by b minus a

so note that this is a generalization of the rolls theorem as if f of a equal to f of b then

so this mean value theorem will write in sort as m v t then mean value theorem implies there exist c in a b says that f prime c is equal to f of b minus f of a

so that is 0 this is equal to 0 that is the rolls theorem

so rolls theorem follows from the mean value theorem but we will prove the mean value theorem using the rolls theorem

so proof first let me explain what this theorem says

so suppose we have this point a and b i have a function which is continuous on this interval a b and it is differentiable in the open interval a b and then let us look at what is f of a

so this this point is a comma f of a this is b comma f of b now if i draw this line joining these two points what is slope of this line slope of the secant line joining a f of a and b f of b is f of b minus f of a divided by b minus a then what the mean value theorem says is that there exist some point c where this slope where the derivative is equal to this slope

so that means that if you see in this picture i have this point c if you look at the slope of this tangent line at this point this is parallel to this line that means that the slope is equal to this slope f b minus f a by b minus a similarly in this picture there is another point here here again the slope is same as this we need to show that there exist c in a b such that the slope of the tangent line at c comma f of c is the same as let me call this slope to be m

so what we want is that f prime c is equal to this slope

so let's see what is the equation

so the equation of the line joining a f of a and b f of b is given by you must have learned in a coordinate geometry that the equation of the line joining two point is given by y minus f of a is equal to f of b minus f of a by b minus a times x minus a

so let us call let us call this line to be l of x is equal to this y which is f of a plus f of b minus f of a by b minus a times x minus a and let g of x be equal to f of x minus minus l of x

so then we have g is continuous on the closed interval a b because f is assumed to be continuous and this l of x is continuous everywhere also g is differentiable on the open interval a b again because f is assumed to be differentiable and l is differentiable everywhere also what is g of a g of a is equal to f of a minus l of a but l is the line joining this point a f of a with b f of b

so l of a is equal to f of a this is equal to if you put x equal to a here l of x is l of a is equal to f of a this is f of a minus f of a which is 0 and g of b is equal to f of b minus l of b what is l of b this is equal to f of b minus l of b is again equal to f of b

so this is also 0 therefore g of a is equal to g of b

so now we have a function g of x which is continuous on the closed interval differentiable on the open interval and g of a is equal to g of b

so we can apply the rolls theorem

so by the rolls theorem there exists at least one c in a b such that g prime of c is equal to zero but what is g of x g of x is f of x minus l of x but g prime

of x is equal to f prime x minus l prime of x and the derivative l prime of x is nothing but the slope of this line

so this is equal to f prime x minus the slope f of b minus f of a by b minus a

so therefore g prime c equal to 0 implies that f prime c is equal to f of b minus f of a by b minus a this is what we had to prove now we will look at some applications of this rolls theorem and mean value theorem

so one is corollary one suppose f from a b to r be continuous and suppose we assume that the derivative f prime x this is equal to 0 for all x in the open interval a b then f must be constant then f must be a constant let x one and x two be any two distinct point in the interval close interval a b we have to show that f of x one must be equal to f of x two but what we know is that by the mean value theorem there exist some c in the open interval x one x two such that f prime c is equal to f of x two minus f of x one by x two minus x one this is because we know that the function is continuous in the closed interval x 1 x 2 and it is differentiable in this open interval therefore by the mean value theorem there is some c such that f prime c is equal to this ratio but f prime x is 0 for all x in a b

so f prime c is equal to 0 therefore f of x two must be equal to f of x one

so this is one application of the mean value theorem that if the function is differentiable and is the derivative is zero on an open interval then the function must be constant in that interval

so in the next class we will see some more applications of the mean value theorem and then we will see some more problems thank you you