

welcome to the next lecture on derivatives in the last lecture towards the end we were looking at the derivatives of function defined in parametric form

so today we will continue with some more examples of derivatives of function defined in parametric form and then we will look at some other results

so for example let us look at equation of the parabola $y^2 = 4ax$ can be written in the parametric form as $x = at^2$ and $y = 2at$ if you put $y^2 = 4a^2t^2$ $y^2 = 4a^2t^2$ square which is equal to $4a^2t^2$

so to find $\frac{dy}{dx}$

so $\frac{dx}{dt} = 2at$ and $\frac{dy}{dt} = 2a$ this implies therefore $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2a}{2at} = \frac{1}{t}$ this is the derivative in terms of the parameter t we could have also calculated directly here directly if we differentiate with respect to x of $y^2 = 4ax$ this gives $2y \frac{dy}{dx} = 4a$ which implies $\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$ putting $y = 2at$ in this we get $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$ which is same as this one lets look at one more example find $\frac{dy}{dx}$ if x is given by $a \cos \theta + b \sin \theta$ and y is $a \sin \theta - b \cos \theta$

so here x and y are given given in terms of the parameter θ

so to find $\frac{dy}{dx}$ we need to find the derivative with respect to θ

so $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ which is equal to $\frac{dy}{d\theta}$ is equal to $a \cos \theta + b \sin \theta$ if you differentiate this $\sin \theta$ gives $\cos \theta$ minus derivative of θ $\cos \theta$ will give by product rule derivative of θ is 1

so this gives $\cos \theta - \theta \sin \theta$ derivative is $\sin \theta$

so this becomes $\sin \theta$ and the derivative of x with respect to θ gives $a \sin \theta + b \cos \theta$

so here $\cos \theta$ cancels and $\sin \theta$ cancels and a cancels

so this is equal to just $\tan \theta$ ok

so then the next thing is we can talk about higher order derivatives

so suppose y is equal to $f(x)$ is differentiable and the derivative $f'(x)$ is a differentiable function then we can find the derivative of $f'(x)$ the derivative of $f'(x)$ is called the second derivative of $f(x)$ and we denote this and is denoted by $f''(x)$ or if i write $y = f(x)$

so the derivative is denoted by $\frac{d^2y}{dx^2}$

so the second derivative is the derivative of the first derivative and similarly we can define higher order derivatives that means the third fourth derivatives also

so for example let y is equal to $a \cos x + b \sin x$ where a and b are constant and

so that if i take the second derivative of y with respect to x this plus y is zero

so we need to find first the first derivative and then differentiate it again to get the second derivative

so $y = a \cos x + b \sin x$ that means that $\frac{dy}{dx} = -a \sin x + b \cos x$ and

so the second derivative $\frac{d^2y}{dx^2} = -a \cos x - b \sin x$ therefore which is equal to simply minus of y

so $\frac{d^2y}{dx^2} + y = 0$ now let us talk about sign of derivatives

so suppose $f(x)$ is an increasing function in an interval lets say i is equal

to some open interval a b

so what does this mean

so that is if x_1 x_2 belongs to I and x_1 is less than x_2 then f of x_1 is less than or equal to f of x_2

so this we call increasing function or non decreasing function if whenever x_1 is less than x_2 then f of x_1 is less than or equal to f of x_2 and we say strictly increasing if we say f of x_1 is strictly less than f of x_2 similarly we can define decreasing function

so here the graph of the function will be if I have this interval a to b in this interval the value of the function keeps on increasing as you go from a to b now what can we say about the derivative what can we say about f' x if f x is a differentiable function

so note that how is the derivative defined

so we have f' of x is nothing but limit of h approaching 0 of f of x plus h minus f of x divided by h

so if we look at the right hand derivative this is equal to the limit of h approaching 0 from the right of f of x plus h minus f of x by h

so here if you see because f is increasing function this f of x plus h greater than or equal to f of x here this is greater than f of x since f is increasing

so the numerator here f of x plus h minus f of x this is non-negative and the denominator h is positive

so therefore the derivative is this must be greater than or equal to 0 if the limit exists then this limit must be non negative similarly if you look at the left hand derivative this is the limit of h approaching 0 from the left of f of x plus h minus f of x divided by h now here because h is negative here both numerator and denominator are negative therefore again left hand derivative is again greater than or equal to zero

so what we have seen is that thus if f x is a differentiable function and is increasing on an interval I then the derivative f' x must be greater than equal to zero similarly for a decreasing function which is differentiable f' x must be less than equal to zero right

so if it is increasing then the derivative is greater than equal to zero if it is decreasing then the derivative is less than equal to zero as an example let us look at f x is equal to x square if you draw the graph of this function we get this parabola y equal to x square and we can easily see that

so this is f x is decreasing in the interval minus infinity to 0 and increasing in the interval zero to infinity right the function is decreasing for x negative and for x positive the function is increasing

so thus f' x is less than or equal to zero on minus infinity to zero and this is greater than equal to zero on zero to infinity of course here I can calculate the derivative here and if I directly calculate the derivative f' x is equal to $2x$ this if x is negative this is less than zero on minus infinity to zero and this is greater than 0 on 0 infinity

so this is an example to show that this function which is decreasing in some interval and increasing in some other interval there the derivative sign is negative when it is decreasing and positive when it is increasing okay next thing i'll discuss is about local minima and maxima of a function

so suppose f x is a given function a point x naught is said to be a local minimum let me also define maximum or local maximum of f of x if there is an interval let me call a b containing x naught such that f of x naught is less than or equal to for minimum this will be less than equal to f x and for maximum this is greatest for all x belonging to a b and f x naught is greater than equal to f x for all x in a b for local maximum

so let me explain this by a graph suppose we have this graph if we see this

point here x naught this is a local minimum of this function because if you see if i take an interval here a b then the value of the function f of x naught is the minimum of all the value of the function in the this interval but if i look at this one

so this is a local min but this point and this point corresponding to this we have these point this is local max this again corresponds to local max because if you see here i can take an interval like this and then you see that this is the maximum value

so here f of x naught is the minimum value in some interval containing x naught and for local maximum f of x naught is the maximum value of f of x in some interval

so if we look at this graph for $f(x) = x^2$ if i look at x naught equal to zero this is a local minimum this is because here from the graph you can see that this value at zero gives me the minimum value in any interval containing zero

so therefore this is local here it is also the global minimum because this is the minimum value of the function but how do we determine this local minimum or maximum using the derivative

so so here if we see local minimum that means what the value of the function to the left of this must be greater than this value and the value of the function to the right must also be greater than that means that the function must be decreasing in the interval to the left of this x naught and it must be increasing in the function in the interval to the right of this point x naught

so x naught is a local minimum if $f(x)$ is decreasing in an interval to the left of x naught and is increasing in an interval to the right of x naught similarly for local maximum it will be the other way for local maximum the function is increasing to the left of x naught and decreasing to the right of x naught and now this we can express in terms of the derivative we have the first derivative test to determine local min or local max of any differentiable function f of x what is this test

so if we have this x naught and to the left of x naught we want for local minimum this should be decreasing

so this we denote like this this is decreasing here and increasing to the right of x naught then this is local min and for local max we must have the function is increasing and to the left and decreasing to the right now if we look at this sign of $f'(x)$ for local min the function is decreasing to the left of x naught that means that the sign of $f'(x)$ is negative to the left of x naught and positive to the right of x naught and for local max this is the other way it is positive to the left of x naught and negative to the right of x naught

so the first derivative test says that if the function is differentiable and if the sign of the derivative changes from negative to positive around x naught then we get local min and if it changes from positive to negative then this must be a local max

so let us look at some example lets look at $f(x) = x^2 - 3x + 2$

so this function if i find the derivative $f'(x)$ this is equal to $2x - 3$ now we want to look at this sign of this $f'(x)$

so $f'(x)$ you can see that $2x - 3$ this is equal to zero at x equal to three by two and this is negative if x is less than three by two this is positive if x is greater than three by two

so we get this point three by two and for x less than three by two the derivative is negative and for x greater than three by two the derivative is positive that means that the function must be decreasing here and increasing to the right of three by two this means that

so therefore by the first derivative test effects has a local minimum at x equal to three by two in fact here i can write this $f(x)$ as this is x^2 minus two times three by two x plus 3 by 2 square and then plus 2 minus 3 by 2 square

so this is equal to $x - 3$ by 2 whole squared and then i have 2 minus 9 by 4 so that gives me minus one by four

so we see that this $x - 3$ by 2 square this has to be always greater than equal to zero

so this $f(x)$ has to be greater than equal to minus one by four and if i put x equal to three by two then $f(x)$ is exactly equal to minus one by four

so f of three by two is equal to minus one by four therefore $f(x)$ takes the minimum value at x equal to three by two and this is the minimum here if you draw the graph this $f(x)$ is equal to $x - 3$ by 2 square minus one by four

so at x equal to three by two this takes the value minus one by four and you can plot this this is a parabola the minimum value is at three by two and if you put x equal to zero this gives me two

so we get a parabola like this and this three by two is the local minimum as well as global minimum in this case ok ah i will state one important theorem this says that any continuous function f of x on a closed interval lets say closed interval a b attains its minimum as well as maximum value on a b

so what this is saying is that if we are have any close interval a b and the function is continuous then there exists that is for continuous f of x on closed interval a b there exists some x_1 and x_2 in this a b such that f of x_1 is always less than equal to f of x and f of x is less than equal to f of x_2 for all x belonging to a b

so we will not look at the proof at the proof of this theorem but note that the assumptions are necessary

so there are two crucial assumptions in this theorem one is that this function is continuous and another is that the interval is closed interval

so first one it is very easy to see that if the function is not continuous so continuity is necessary because otherwise you can have say this is the function and suppose i take and at this point this is equal to zero

so the function is defined on the closed interval zero one this is $f(x)$ is equal to x for zero less than equal to x less than half and this is equal to zero at x equal to half and then this is one minus x if x is bigger than half and less than equal to one this function you can see that this is discontinuous at half now if you look at this function this does not attains its maximum value

so no maximum value another thing is that closed interval is again necessary the result is false for open interval for example consider $f(x)$ equal to one by x on the open interval zero one

so the function is one by x note that this function as x goes to zero this goes to positive infinity

so again this function $f(x)$ has no maximum value on the open interval zero one even though it is continuous

so we require both the continuity as well as the interval being closed for this theorem to be true

so after this the next we will learn the two other very important theorems on derivatives which are the rolls theorem and the mean value theorem

so let me first state the rolls theorem

so this says that suppose $f(x)$ is a function defined on a b satisfying the following three conditions first one is that we require f of x to be continuous on the closed interval a b this is on closed interval a b second is $f(x)$ is assumed to be differentiable on open interval a b and the third condition is

that the value of the function at the end point $f(a)$ is equal to $f(b)$ then the conclusion is that then there exists at least one point c belonging to the open interval (a, b) such that $f'(c)$ is equal to zero

so let me try to explain this theorem by showing a picture

so let me explain this rolls theorem by means of these examples

so i have this interval (a, b) and what we have the third condition says that $f(a)$ must be equal to $f(b)$ and the function is continuous in this interval and differentiable in the open interval

so it might be that we have a function like this or it can be i have $f(a)$ and $f(b)$

so if you see what the conclusion says that there exists at least one point c where the derivative is equal to 0 and we know that the derivative equal to 0 means that the slope of the tangent line is parallel to the x axis

so here if you see this point the tangent line is parallel to x axis here also

so there are two values here where the derivative is 0 here again we have a value c here where the derivative is 0 .

so what this theorem says is that no matter what the function is if it satisfies these three conditions that it must be continuous in the closed interval $[a, b]$ differentiable on the open interval and $f(a)$ is equal to $f(b)$ then we must have that at some point in between a b the derivative must be zero ok

so this rolls theorem we will again try to see that these conditions that are specified here these are necessary conditions

so we will not look at the proof right now but we will show that the conditions are necessary

so first one that we said that the function must be continuous on the closed interval suppose we have this example i have a function like this and then i define let us say this is one and four and i defined the value of this function is equal to this

so this function is $f(x) = 4$ if $x = 1$ and this is equal to x if $1 < x \leq 4$

so this function if you see this $f(x)$ is not continuous at $x = 1$ but other than that the function is continuous everywhere other than $x = 1$ $f(x)$ is continuous everywhere else also this function $f(x)$ is differentiable on the open interval $(1, 4)$ and $f(1)$ is equal to $f(4)$ but if you see this function there is no point where the derivative is equal to zero there is no point in the open interval $(1, 4)$ but $f'(x)$ because $f(x)$ is equal to x in the open interval $(1, 4)$ $f'(x)$ is equal to one for all x belonging to $(1, 4)$ thus there is no c in $(1, 4)$ for which $f'(c)$ is equal to zero however this example does not contradict the rolls theorem because $f(x)$ is not continuous on the closed interval $[1, 4]$ all right

so i will stop here today in the next lecture i will show that the other two assumptions the second assumption on the differentiability of the function in the open interval (a, b) and the third assumption that $f(a) = f(b)$ are also necessary for the conclusion of the rolls theorem and then we will discuss the mean value theorem and then some applications of these theorems thank you you