welcome to the next lecture on derivatives

so in the last lecture we studied about the exponential function and then we calculated the derivative of the exponential function today we will learn about the logarithmic functions which is the inverse of the exponential function and then some other things let's start by recalling exponential function

so this is written as exponential of x or e to the x this is a function whose domain is

so this is from r to r plus which is same as the positive real line

so domain is the set of real numbers ranges all positive real numbers and also we have that the exponential function this is strictly increasing function that means that if x one is less than x two then e to the x one is less than e to the x two

so from this it follows that

so this function x going to e to the x is an injective which is also called one to one function injective function from the set of real numbers on to the set of positive real numbers r plus now suppose f is any function from some set x to y is and one to one and onto function then we can define the inverse function which is denoted by f inverse this is a function whose domain is y and the co domain is x

so we have inverse function f inverse from y to x this is simply taken as

so we have f inverse of y is equal to x if and only if f of x is y

so to know the inverse of y we need to look at the value of x for which f of x is y right and because this is a one to one and onto function we know that there is a unique x for every x here we know there is a y in y and because this is one to one onto function the image of two different x cannot be the same y

so now for every y here we look at the x says that f of x is equal to y and then f inverse is the map from y to x

so this is the general thing that you must have learnt in function that inverse of a function can be defined if we have n one to one and onto function

so now for f x is equal to e to the x we know that this is an one one onto function from x is the set of real numbers on to the set of positive real numbers we can define the inverse of exponential function which is called the logarithmic function and is denoted by l n of x

so this is also called the natural logarithm of x

so log of x is nothing but the inverse of exponential x

so if i write y is equal to ln x this is equivalent to x is equal to exponential of y now what are the properties of logarithm of x

so first thing what is the domain domain is r plus that is ln x is this is not defined for x negative or zero

so for x less than equal to zero this is not defined l n x is defined only for all positive real numbers range of l n x is the set of all real numbers this is because the exponential is defined from r to r plus and another property is that this ln of x is also an increasing function of x

so if x one is less than x two then l n x one is less than l n x two and just like we had the limit

so what is the limit as x approaches positive infinity of l n x this is equal to positive infinity and the limit as x approaches 0 from the right side of ln x this is equal to negative infinity this is because limit of e to the x as x approaches positive infinity is positive infinity and limit of x approaching negative infinity of e to the x is equal to zero therefore the limit as x approaches zero from the positive side of l n x is negative infinity let us try to draw the graph of l n x and let me also draw the graph of e to the x

so the exponential function we have seen that the graph looks like this at x equal to zero this is 1 and as x increases e to the x keeps on increasing goes

to infinity as x goes to infinity and it goes to zero as x goes to negative infinity now what about log of x this is the inverse of for exponential x

so the inverse of a function the graph of the inverse of a function can be plotted by taking the mirror image of this is the line y equal to x

so you look at the mirror image of the graph of f of x in the line y equal to x that gives the graph of f inverse of x

so what happens is this is the graph of log of x looks like this and the value here is one

so ln of one is equal to zero this is because we know that exponential of zero is one

so this passes through the point one comma zero and as x increases this keeps on increasing note that ln of x this is positive if x is greater than 1 and ln of x is negative if x is less than 1 and of course ln x is defined only for positive x

so it is undefined for x less than or equal to 0 and this goes to negative infinity as x is going to 0 from the right previously in school you might have learnt logarithmic functions

so let me compare this ln x with you might have learnt log of x before

so how is log of x is defined

so recall that if i write y is equal to log of x this is equivalent to saying that x is equal to 10 raised to power y

so to find the log of anything you write the number as the exponent of ten

so for example if i ask you what is log of 100

so this is equal to 2 because 10 to the power 2 is equal to 100

so this is analogous to we also here we saw that y equal to ln x this is same thing as x is equal to e to the power y

so instead of 10 we are using e here in general we can define logarithm to the base b by if i write y is equal to log x to the base b this is if and only if x can be written as b to the power y and we take this b to be any positive real number and b is not equal to one this is because if i take b equal to one then one to the power y is always equal to one

so if you take the function 1 to the power x that is the constant function

so we cannot define the inverse of that but if b is any positive real number other than 1 then b to the y can be shown to be a 1 to 1 function and then the inverse is the logarithm of the function to the base b

so formally a to the power x for a bigger than 0 can be defined using the exponential function e to the x as follows

so a to the x is nothing but the exponential of x ln a ok

so the exponential function e to the x we have seen what it is and for any other the exponent we can define a to the x is equal to e to the x ln a note that ln a is well defined as a is a positive real number ok now lets see some other properties of log

so one is that the logarithm of product x times y is equal to sum of the logarithm and logarithm of x by y this is equal to ln x minus ln y this is if x is greater than 0 y is greater than 0 again here x greater than 0 y greater than 0 and logarithm of x to the power any m this is equal to m ln x here again x is positive

so these properties you must have seen for the common logarithm to the base 10 and these are true for the natural logarithm also this can be proved because to show this you have to show that the exponential of the left hand side is equal to the exponential of the right hand side

so proof of one say

so let a is equal to l n x and b is equal to l n y then what is x x is equal to e to the power a and y is equal to e to the power b and then we have to find

what is ln of x times y

 so what is x times y x times y is e to the power a times exponential of b and we have seen that the exponential has the property that exponential of a times exponential b is equal to exponential of a plus b and therefore what is ln of xy is equal to a plus b the logarithm of natural logarithm of any quantity is the exponent that occurs in e to the power this

 so this is same thing as a is l n x b is l n y similarly others can be proved another property which is important is suppose you have some other base

 so if i write log of x to the base b this i can express in terms of the natural logarithm

 so this can be written as ln x divided by ln of b and note that this is defined for b not equal to one

 so ln b is non zero note that ln b is not equal to zero since b is not equal to one

 so any logarithm to any base can be converted into natural logarithm

 so it is enough to study natural logarithm and then we can deal with logarithm to any base b

 so this again one can prove suppose let me write a is equal to log of x to the base b and let us write m is equal to ln x and n is equal to ln of b then x is equal to b to the power a also x l n x is equal to m

 so x is equal to e to the power m and b is equal to e to the power n now we have x is equal to b to the power a

 so we have e to the m which is equal to x and x is equal to b to the power a but b is nothing but e to the power n raise to power a which is equal to e to the power n a this implies m must be equal to n a because e to the x is a one to one function

 so m is equal to n a and then what we had to prove is that this quantity was a is equal to log x by log b that is m by n

 so this implies a is equal to m by n that is a is log x to the base b is equal to ln x divided by ln of b

 so now we will try to find the derivative of ln x

 so remember that we have seen that using chain rule we can calculate the derivative of the inverse of any function if i know the derivative of the function

 so we will use that to calculate the derivative of ln x

 so let y is equal to l n of x

 so we want to find d y by d x but we know that this implies that then x is e to the power y and because we know that we know the derivative of exponential function

 so we can differentiate this expression with respect to x

 so d by d x of x is equal to d by d x of e to the y and this implies this is one is equal to now here we use chain rule

 so this is d by d y of e to the y times d y d x by chain rule and the derivative of e to the y is same as e to the y times d y d x but e to the y is equal to x

 so this can be written as x times d y d x and this implies d y d x is equal to 1 by x

 so what we have got is the derivative d by d x of the function natural log of x is equal to 1 by x and because ln x is defined only for x positive this formula is for all positive real number x we can define also if we replace x by mod of x the absolute value of x then ln of mod x this is defined for all x except for 0 because mod x is always a non negative real number

 so if x is non zero then mod x is always positive

 so log of mod x can be defined can we calculate the derivative what is the

derivative of f x equal to log of mod x

so if we see f x is log of mod x this can be defined piecewise as this is equal to ln of x if x is greater than 0 and this is equal to ln of minus x if x is less than 0 because for x greater than 0 mod x is equal to x and for x less than 0 mod x is equal to minus x

so this is the function now we know the derivative of ln x

so for x greater than 0 zero f prime x is equal to one by x we have seen and for x less than zero f prime x is d by d x of l n of minus x now this again we can use chain rule and this is equal to the derivative of ln of minus x with respect to minus x which will be one by minus x times the derivative of minus x with respect to x d by d x of minus x which is minus one

so this is again equal to one by x

so d by d x the derivative of log of mod of x is also equal to one by x this is true for x belonging to any real number except zero

so when you learn about anti-derivative or the indefinite integral there you will see that in the formula the integral of the anti-derivative of 1 by x is written as log of mod x and not just log of x

so ok now let us look at a few examples

so now we know the derivative of log x also

so we can look at some example involving log

so suppose f x is equal to sine of l n x then what is f dash x the derivative we use chain rule

so derivative of sine gives me cosine cosine log x and then the derivative of log x with respect to x gives me one by x

so this is true and of course this is defined for every positive x just look at one more g of x is equal to cosine of log x plus e to the x again this function is defined for all x greater than zero then g prime x again we use chain rule the derivative of cosine will give me negative sign of log x plus e to the x times the derivative d by d x of inside function is log x plus e to the x and then this is equal to minus sine log x plus e to the x the derivative of sum is sum of the derivatives

so d by d x of log x gives me one by x plus d by d x of e to the x is e to the x

so this is the derivative of g of x ah lets also calculate the derivative of a to the x where a is positive real number

so lets see a to the x we know is that this is defined as the exponential e to the power of x ln a

so d by d x of a to the x is equal to d by d x of e to the x ln a and now we use chain rule the derivative of this will be equal to e to the x ln a times the derivative d by d x of x ln a now here natural log of a is a constant

so d by d x of x times ln a is simply equal to ln a

so this is equal to e to the x ln a times ln of a and e to the x ln a is equal to a to the x

so d by d x of the derivative of a to the x is equal to a to the x times natural log of a you can see that in particular if i put a equal to e then ln of e is equal to 1 this gives me the usual formula d by d x of e to the x is equal to e to the x there is another way of also doing this

so what you do is that you write y is equal to a to the x now take natural log both sides this implies l n of y is equal to x times ln of a this is because ln of a to the x we know is x times ln a and now differentiate this with respect to x therefore d by d x of l n y is equal to d by d x of x ln a now here because y is a function of x i can use chain rule this implies this is equal to 1 by y times d y d x is equal to l n a which implies d y d x is y times ln a which is a to the x ln a

so this gives the same answer but it might be slightly easier for you to write in this form ok

so next we will see that the second way of doing this can be done more generally and we will discuss what is called the logarithmic differentiation

so suppose we have a function f of x which can be written as some function u of x raised to power v of x

so previously it was y is equal to a to the x here u of x is just the constant a and v of x is the function x but now we are allowing both these to be a function of x and we want to find f prime x we want to find the derivative f prime x then we do the same thing as we did for the previous example

so we take natural log to both sides

so we have l n of f x is equal to l n of ux to the power vx and we know by the property of logarithm this is same thing as vx ln of ux and now we differentiate both sides with respect to x

so differentiating with respect to x we get the derivative of natural log of f x will give me 1 by fx times f prime x this is again by the chain rule and then the derivative of vx times ln of ux here i can use product rule this is equal to d by dx of vx ln ux and this is equal to v prime x times natural log of u x plus v x times the derivative d by d x of l n of u x this is by the product rule and then again we use chain rule to find the derivative of l n u x is 1 by u x times u prime x

so this is equal to v prime x ln of u x plus v x times 1 by u x times u prime x

so this implies f prime x is equal to f x times this quantity here that is v prime x l n u x plus v x by u x times u prime x and f x is nothing but u x to the v x

so we get the derivative of f of x in terms of x

so let us look at some examples

so first one we will look at f x is equal to x to the power of sine x then we take log ln f x is equal to ln of x to the sin x which is equal to sine x ln x and then we differentiate this this implies one by f x times f prime x is equal to the derivative of this will give me cos x log x and then plus sine x times derivative of log x is one by x this implies f prime x is equal to f x which is x to the sine x times cos x log x plus sine x by x

so note that you do not need to remember the previous formula for the derivative of f x you can just follow the steps and then calculate the derivative lets see another example

so here what we have is now we have in an implicit equation

so find d y d x if y to the x plus x to the y is equal to one

so now here note that we cannot directly take log of this but separately if you see these are the exponents

so what we can do is that you let u to be the function y to the x and v is equal to x to the y then we can calculate what is d u by d x then what is given is that then the given equation becomes u plus v is equal to 1 therefore if i differentiate this d u d x plus d v d x this is equal to zero let me call this equation one u is equal to y to the x this implies l n u is equal to x l n y which implies if i differentiate with respect to x i get one by u times d u by d x is equal to the derivative of this with respect to x derivative of x with respect to x gives me one times l n y plus x times the derivative of l n y with respect to x

so that is 1 by y times d y d x and this implies d u d x is equal to u which i can again write as y to the x times l n y plus x by y d y d x let us call this equation two

so since v is equal to x to the y ln of v is equal to y ln x and now we differentiate this to get 1 by v dv dx is equal to the first term here is y

so dydx times ln x plus the first term times the derivative of second term ln x gives 1 by x

so this implies dv dx is v which is x to the y times this is ln x d y d x plus y by x this is equation three now using two and three in 1 we get the value of d u d x and d v d x we get y to the x ln y plus y to the x times x by y d y d x this is d u d x plus d v d x gives me x to the y ln x d y d x plus x to the y times y by x this is equal to zero now we want to calculate what is d y d x

so these two terms we can combine this implies and then you see we have y to the x divided by y which can be written as y to the x minus one

so this is x times y to the power x minus 1 plus the other term is x to the y ln x x to the y ln x times d y d x this is equal to the negative of y to the x ln y plus the other term is x to the y y divided by x

so this is x to the y minus one times y

so this gives d y d x is equal to negative of y to the x l n y plus x to the y minus one times y divided by x to the y minus 1 y to the x minus 1 plus x to the y ln x right

so we have calculated this as an exercise you can try one is cos x to the power y is equal to cos y to the power x find d y d x another is find d y d x f y to the x plus x to the y plus x to the x is equal to a to the b where a and b are constants right

so both these you can do in the sim in a similar way you let this to be u this is equal to v then you calculate here you can even directly take logarithm both sides and then you can differentiate it but in the second example you can take these three terms to be u v and w and then find what is d u d x d v d x and d w d x and then we know that the sum is a constant

so the sum of the derivative will be zero and from that you can calculate what is d y d x

so this logarithmic differentiation is an important tool to calculate the derivative of functions when it looks complicated to calculate another thing we will learn about is what is called the derivative of functions when x and y are given in some parametric form

so we want to calculate derivatives of functions in parametric form

so let us take let us assume that x and y can be written in terms of some parameter lets call t that is x is a function of t and y also is a function of t then we want to find d y d x

so to find d y d x if you look at since y is a function of t if i write d y d t this can be written as d y d x times d x d t this is by chain rule and therefore this implies that the derivative d y d x can be calculated in terms of the derivatives with respect to t

so this is the formula for the derivative of y with respect to x in terms of the derivatives with respect to t

so d y d x is equal to d y d t by d x d t or this i can also write in as y prime t by x prime t

so you can calculate the derivative by finding the derivatives with respect to t rather than trying to solve for y as a function of x

so as an example if i take the equation the equation of circle x square plus y square is equal to a square can be parameterized as x is equal to a times cosine t and y is equal to a sine t because we know that cosine square t plus sin square t is one

so x square plus y square is a square now if i want to find what is derivative d y d x

so we know that d x d t is minus a sin t and d y d t is equal to a times cosine t

so therefore to find derivative d y d x this is nothing but d y d t by d x d t

which is equal to a cosine t by negative a sin t
 so a cancels and i get negative of cotangent of t right
 so we get the derivative here in terms of the parameter t
 so that finishes today's lecture in the next lecture we will see some more
examples of derivatives in terms of a parameter and then we will look at some of
the applications of derivatives thank you you