

hello students

so today is the fourth lectures lecture on derivatives in the last lecture we looked at the chain rule for differentiation and then we were looking for derivatives of inverse trigonometric functions in the last class we calculated the derivative of sine inverse of x and tan inverse of x and then i said that similarly we will calculate the derivative of other inverse trigonometric functions in the next class

so we saw before that d by dx of the derivative of sine inverse of x is equal to 1 over square root of 1 minus x square and derivative of tan inverse of x is one over one plus x square now in the lecture of trigonometry you must have learnt some identities

so we know that sine inverse x plus tan inverse x plus cosine inverse x is equal to π by two and tan inverse x plus cotangent inverse x is also π by two and similarly for secant inverse x plus cosecant inverse x is π by two

so once i know the derivative of sin inverse x i can calculate the derivative of cos inverse x using this identity

so therefore cos inverse x is nothing but π by 2 minus sin inverse x this implies that the derivative of cos inverse x is equal to the derivative of π by 2 is 0 minus d by dx of sine inverse x

so we get this is equal to minus 1 over square root of one minus x square and similarly derivative of cot inverse x is equal to minus of derivative of tan inverse x which is minus one over one plus x square now we are left with the derivative of secant inverse x and cosecant inverse x

so if i can calculate the derivative of secant inverse x then again the derivative of cosecant inverse x will be negative of the derivative of secant inverse x

so lets calculate the derivative d by dx of secant inverse x

so we will do the same way that we did for sin inverse and tan inverse let y is equal to secant inverse x then x is equal to secant of y

so if i differentiate with respect to x we get hand side the derivative of x is one is equal to d by dx of secant of y and by chain rule this is equal to d by dy of secant y times dy dx this is by chain rule but what is the derivative of secant y we know this is equal to secant of y times tan y times dy dx this implies the derivative dy by dx is equal to one over secant y times tan y we want to express this derivative in terms of x y was secant inverse of x

so secant of y is since secant of y is equal to x i can replace secant y by x here and what about tan square of y we know is secant square y minus 1

so this is equal to x square minus 1 this implies tan y is plus or minus square root of x square minus one now we have to determine whether this is positive or negative sign

so recall from your trigonometry lectures that secant inverse of x if i write that is my secant inverse of x this belongs to 0 to π by 2 if x is greater than 1

so and this is in π by two to π if x is n minus infinity to minus one

so this is because the secant inverse of x is always between zero to π and it can never be π by 2 and if x is bigger than 1 then bigger than or equal to 1 then secant inverse x is less than π by 2 and greater than equal to 0 and if x is less than equal to minus 1 then secant inverse x is greater than π by 2 and less than equal to π and secant inverse x is not defined for secant inverse x is not defined if x is between minus one and one this is because secant of theta is always greater than equal to one or less than equal to minus one

so therefore y is equal to secant inverse x this we got that this is in 0 to π by 2 if x is between one and infinity and this belongs to π by two to π if x is less than equal to minus one now what we want is we want to find what is tan

of y we know that y is in the interval zero to π by two for x greater than equal to 1 and y is in the interval π by 2 to π if x is in minus infinity to minus 1.

so this implies \tan of y this is greater than equal to zero if x belongs to one infinity and this is less than or equal to zero if x is in minus infinity to minus one this is because we know that \tan function is positive in the first quadrant and negative in the second quadrant

so therefore if you see in the previous page we had $\tan y$ is plus or minus square root of x square minus 1 if we have x to be greater than equal to 1 then \tan of y must be non negative

so therefore \tan of y is equal to square root of x square minus 1 if x is greater than equal to 1 and this is minus of square root of x square minus 1 if x is less than equal to minus 1 this was \tan of secant inverse of x

so therefore d by $d x$ of secant inverse x is equal to 1 by secant y is equal to x let me write again secant y times $\tan y$ which is equal to 1 by x times square root of x square minus 1 if x is greater than equal to 1 and this is equal to 1 by minus x square root x square minus 1 if x is less than equal to one because \tan of y is minus square root x square minus one if x is less than equal to minus one sorry

so we can combine this and see that if x is greater than equal to 1 then we have positive sign x and for x less than equal to minus 1 minus x is again positive minus x is equal to $\text{mod } x$ since $\text{mod } x$ is equal to x if x greater than equal to 0 and minus x if x is less than 0 we can write d by $d x$ of secant inverse x is equal to 1 over $\text{mod } x$ times square root of x square minus one and of course we know that this is defined if $\text{mod } x$ is greater than or equal to one

so this is the formula for derivative of secant inverse x you should remember that here we have one over $\text{mod } x$ times square root of x square minus 1

so if x is negative less than or equal to minus 1 then this will become the derivative will be negative and hence we also get derivative of cosecant inverse x this is equal to minus of derivative of secant inverse x

so minus 1 by $\text{mod } x$ times square root of x square minus one again this is defined only for $\text{mod } x$ greater than equal to one

so we have used chain rule and proved the derivative of all inverse trigonometric functions now the next thing we will see is to find the derivative of a function y is a function of x where we have y is defined as an implicit function in x

so let me do the implicit differentiation

so sometimes we have an equation relating y to x but it is difficult to write explicitly y as a function of x for example suppose we are given y plus sine y is equal to x

so here we know that y depends on x but it is not possible to write y directly as a function of x and we want to calculate $d y$ by $d x$

so our aim is to find the derivative $d y$ by $d x$

so here we cannot write here instead of trying to write y as a function of x we will do implicit differentiation as follows

so what we do is that we simply write the derivative

so y plus sign y is equal to x this implies that if i take the derivative d by $d x$ of y plus sine y this is equal to d by $d x$ of x now this gives this is $d y$ by $d x$ plus the derivative d by $d x$ of sine y is equal to one now by chain rule we know that the derivative of sine y with respect to x can be written as d by $d y$ of sine y times $d y$ by $d x$ and here we are using chain rule

so this implies that i can take $d y$ by $d x$ common times one plus the derivative of sine y gives cosine y is equal to one this implies $d y$ by $d x$ is equal to one over

one plus $\cos y$ provided $\cos y$ is not equal to minus one

so note that by doing implicit differentiation we need not get the derivative $\frac{dy}{dx}$ as a function of x only

so in this example the derivative $\frac{dy}{dx}$ is $\frac{1}{1 + \cos y}$ now we do not know $\cos y$ in terms of x directly

so in general the derivative $\frac{dy}{dx}$ will be a function of x and y and we know that y implicitly is a function of x by this equation

so so that is about implicit differentiation now the next thing that i would like to do is

so far we have considered the functions which are polynomials or trigonometric functions or inverse trigonometric functions or some x to the some power but there are also other functions which are very useful in calculus

so like the exponential functions and logarithmic functions

so i would like to introduce you the exponential and logarithmic function and then see what the derivatives are

so let us talk about exponential function

so we will define exponential of x this is equal to $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$ up to infinity

so this is what this is nothing but this is equal to this we also write as summation of n equal to zero to infinity of $\frac{x^n}{n!}$ and which is equal to the limit of summation $\frac{x^k}{k!}$ k equal to 0 to n as n approaches infinity

so this is called the power series that we write a function as an infinite series and any infinite series we can write as the limit of this finite series right

so we will not go into much rigor here but let me write as a fact that this series the above series converges and converges means that to a finite real number for every real number x

so the exponential function is exponential of x is the limit to which this series converges

so thus exponential x is defined for every real number x ok

so further exponential x satisfies the following properties one is what is exponential of zero if you see exponential of x is defined as $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ and

so on if i put x equal to zero only the first term is one and all the other terms are zero

so it is easy to see that exponential of zero is equal to one second thing is that exponential of x is an increasing function of x that is $x_1 < x_2$ must imply that exponential of x_1 is less than exponential of x_2

so this is because if you see here if i take x_2 to be bigger than x_1 then of course $\frac{x_2^2}{2!}$ is bigger than $\frac{x_1^2}{2!}$ and $\frac{x_2^2}{2!}$ is bigger than $\frac{x_1^2}{2!}$

so each term in this power series is greater than the each term in the power series for exponential x_1

so this is an increasing function of course we know that this domain i said that domain of this function is the set of all real numbers

so this exponential x is defined for all r and the range of exponential function this is equal to all positive real number will denote it by r plus which is equal to 0 to infinity

so the exponential x never takes negative value or zero value exponential x is never zero or negative the fifth property i will write is what happens to this exponential x as x approaches positive infinity if you see exponential of x is $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ and

so on

so if x approaches positive infinity each of the terms except for the first term 1 approaching positive infinity

so therefore this limit must be equal to positive infinity and the other one which is not very obvious here but let me write the limit of exponential of x as x approaches negative infinity this is equal to zero

so the seventh property is very important this says that exponential of the sum of x and y is equal to product of the exponential exponential x plus x times exponential y compare with a to the m plus n is equal to a to the m times a to the n if m and n are say natural numbers using seven and one we get exponential of minus x is equal to one by exponential of x this is because we know that 1 is exponential of 0 which i can write as exponential of x plus minus x and by property seven this is exponential x times exponential of minus x and therefore exponential minus x is one over exponential x

so now you can see from this six also follows because if x goes to minus infinity six follows from five because as x goes to minus infinity minus x goes to positive infinity and this exponential of x you can write as one over exponential of minus x that goes to positive infinity and one over infinity that goes to 0.

so let us say draw the graph of this function

so we know that exponential x is defined for all x for x equal to 0 this is 1 so i have 0 comma 1 and this is a increasing function

so as x increases from zero this will keep on increasing and it goes to infinity as you go to positive infinity and if x is negative then because this is increasing function exponential of x would be less than exponential of 0 which is 1 and as you go to negative infinity this goes to 0 right

so this is the graph of the function exponential x this is always increasing it goes to negative infinity it goes to 0 as x goes to negative infinity it goes to positive infinity as x goes to positive infinity now if we put x equal to 1 we get exponential of 1 this is equal to one over i put x equal to one

so one over one factorial plus x square is again one over two factorial and

so on this i can write as this is also summation of one over k factorial k is running from zero to infinity we know that this is some real number exponential of one is a real number and we denote this by e this is called the euler's constant

so e is exponential of one which is equal to summation of one over k factorial k equal to zero to infinity in fact we can show that this e is some real number which is bigger than two and less than three in fact e is approximately equal to two point seven one eight

so though this is not important for us right now in calculus but let me try to show to you why e is bigger than two and less than three

so we know that e is equal to one over one plus one over one factorial plus one over two factorial dot dot dot up to infinity this is of course bigger than one plus one over one factorial which is equal to two

so e is bigger than two why is this less than three also if i write e as one plus one over one factorial plus one over two factorial plus one over three factorial and

so on we can write this as this is less than one plus this is again 1 plus 1 over 2 factorial is same as 2 plus 3 factorial is 3 times 2

so this is less than 1 over 2 square and then let me write one more time 1 over 4 factorial 4 factorial is 4 times 3 times 2 that is bigger than 2 times 2 times 2 which is 2 cube

so 1 over 4 factorial is less than 1 over 2 cube and

so on this is because if i write one over n plus one factorial this is less than one over two to the n for n greater than equal to two this you can prove by induction or directly like i explained now we see that here we are getting a geometric series 1 plus half plus half square and

so on and this geometric series you might have seen that we can sum this infinite series

so let let me write this recall that geometric series a plus a r plus a r square and

so on up to infinity this is equal to a over 1 minus r if the geometric ratio r is less than 1 in absolute value

so this you would have seen in this sum of the geometric progression that as you take the limit for infinite series as long as the common ratio is less than 1 in absolute value this converges and this is equal to a over one minus r

so putting a equal to one and r equal to two gives one plus sorry r equal to one by two

so a r is one by two plus one by two square and

so on this is equal to a is one by one minus one by two which is equal to

so this series from 1 plus half plus 1 4th and 1 8 this sums to 2 and then i have 1 plus this

so therefore e is less than one plus the geometric sum was two which is equal to three i will just state as a fact that it can be proved that e is an irrational number also let me write some limits

so limit of n going to infinity of one plus one by n to the power n if you look at the limit of this sequence 1 plus 1 by n to the power n this gives us exactly equal to e and we can also write the limit of x going to 0 of 1 plus x raised to power 1 by x this is also equal to e

so this fact i will not prove right now another thing that we will need is that let us calculate this the limit of h going to 0 of exponential of h minus one over h

so let me use this notation

so notation we write exponential x as e to the power x also ok

so if i look at we know that exponential of h this is equal to one plus h over one factorial plus h square over two factorial this infinite series

so this implies exponential h i'll write e to the h e to the h minus 1 this is equal to h plus h square by factorial 2 h cube by factorial three h to the n by factorial n and

so on this implies that e to the h minus one over h is equal to one plus h by factorial two plus h square by factorial three h to the n minus one by factorial n and

so on for any non zero h and then it can be shown that the limit of e to the h minus one over h this is equal to one right

so formally you can see that as h approaches zero all these terms h square by factorial 3 and

so on all these terms approaches 0 and the first term is 1

so it can be shown that this limit as h goes to zero it approaches one

so this is one important limit that we will need the limit of exponential h minus one over h is equal to one now i will try to calculate the derivative of e to the x note that e to the x is this exponential function

so let us write f x equal to e to the x then to find f prime x we need to see whether this limit as h goes to zero of f of x plus h minus f of x over h if this limit exists then the limit is the derivative and this is equal to limit of h going to 0 e to the x plus h minus e to the x over h we know that exponential of x plus h is nothing but exponential x times exponential of h minus exponential x by h

so we can take e to the x common this is equal to e to the x times the limit of h going to 0 of e to the h minus 1 over h and this limit we said is equal to 1

so this is equal to e to the x

so e to the x exponential function is a very special function whose derivative is itself

so we got that the derivative d by $d x$ of e to the x is e to the x itself now once we know the derivative of e to the x we can try to calculate some derivatives involving exponential function what is d by $d x$ of e to the power five x

so we know that we can use the chain rule and if i write this as d d five x of e to the five x times d by $d x$ of five x this the derivative of exponential function will give me e to the five x times five what is the derivative of e to the x square this is equal to derivative of exponential is itself but then i have to differentiate by chain rule x square this is equal to two x

so i get two x times e to the x square lets do derivative d by $d x$ of \tan inverse of e to the x we know that the derivative of \tan inverse of x is 1 over 1 plus x square and then times the derivative of e to the x

so this is equal to e to the x divided by one plus e to the two x ok

so this finishes today's lecture in the next lecture we will show that in the next lecture we will define the inverse of exponential function which is called the logarithmic function and then we will let calculate the derivative of \log of x and then we will see some properties of logarithm and then calculate some more derivatives using these functions thank you you