

hello students

so we will continue our discussion with of derivatives

so in the previous lectures we have learnt some of the properties of derivatives like the product rule quotient rule etc and in this lecture we will learn a very important rule which is called the chain rule of derivative

so let us start with the chain rule

so first of all let us recall what is a composition of two functions

so we denote suppose f and g are two functions when we denote by $f \circ g$ this is defined as $f(g(x))$ right

so if we have two functions f and g and the range of g is contained in the domain of f then the composition is defined and it is nothing but $f(g(x))$

so what we would like to know is suppose we know g is differentiable at some x equal to a

so let me write $g(x)$ is differentiable at $x = a$ and the function $f(g(x))$ is differentiable at $x = g(a)$ then is the composition $f \circ g$ differentiable at $x = a$ and if

so then can we write the formula for $(f \circ g)'(a)$ this is what

so let us try to see

so let $h(x)$ is the composite function this is $f(g(x))$ then to check the derivative

so to check whether $h(x)$ is differentiable at $x = a$ we need to look at the limit h going to zero of let me write this composite function as $k(h)$

so $k(h) = f(g(a) + h) - f(g(a))$ divided by h that is same thing as limit of h going to zero $k(h) = f(g(a) + h) - f(g(a))$ divided by h now if $g(a) + h - g(a)$ is non zero in some open interval containing sorry

so let me write like this

so if $g(a) + h - g(a)$ is non zero for all small h is non zero then we can write $f(g(a) + h) - f(g(a))$ by h this can be written as $f(g(a) + h) - f(g(a))$ divided by $g(a) + h - g(a)$ and times $g(a) + h - g(a)$ divided by h

so this we can do if this $g(a) + h - g(a)$ is non-zero now what we know is that we are given that g is differentiable at a

so the limit of h going to 0 of $g(a) + h - g(a)$ by h this is this exists and this is the derivative $g'(a)$

so this is the second factor here in the product also f is differentiable at $g(a)$

so limit of h going to 0 of $f(g(a) + h) - f(g(a))$ by $g(a) + h - g(a)$ this is equal to the derivative of f at $g(a)$ this is because as h goes to zero $g(a) + h - g(a)$ goes to $g(a)$ this is since g is continuous at a we have seen that if g is differentiable at a then it is also continuous

so this denominator goes to 0 and then this limit is same as writing as $f(y) - f(g(a))$ divided by $y - g(a)$ and i can write this y going to $g(a)$ and this we know is equal to $f'(g(a))$

so we have got this formula

so if $g(a) + h$ is not equal to $g(a)$ for small h then $f'(g(a))$ sorry then $(f \circ g)'(a)$ this is equal to $f'(g(a)) \cdot g'(a)$

so the chain rule says that this is true even if we do not impose this condition

so in fact the above formula is always true

so let me write this theorem chain rule

so let f and g be two functions such that f is differentiable at $g(a)$ and g is differentiable at a then $f \circ g$ is differentiable at a and the

derivative is given by $f \circ g'$ of a is equal to f' of g of a times g' of a

so let us first see the proof of this

so we mean we need to show that $\lim_{h \rightarrow 0} \frac{f(g(a) + h) - f(g(a))}{h}$ is equal to $f'(g(a)) \cdot g'(a)$

so what we will do is because we do not know whether we can divide by $g(a) + h - g(a)$

so we define a new function ϕ of y this is equal to $\frac{f(y) - f(g(a))}{y - g(a)}$ this makes sense only if y is not equal to $g(a)$ and if y is equal to $g(a)$ we can define this as $f'(g(a))$ we know that $f'(g(a))$ exists

so ϕ of y is a function defined in this form if y is not equal to $g(a)$ this is $\frac{f(y) - f(g(a))}{y - g(a)}$ and it is equal to $f'(g(a))$ if y equal to $g(a)$ now one important thing is that this function then becomes continuous

so if we look at $\lim_{y \rightarrow g(a)} \phi(y)$ this is equal to $\lim_{y \rightarrow g(a)} \frac{f(y) - f(g(a))}{y - g(a)}$ now because it is given that f is differentiable at $g(a)$ this difference coefficient this goes to $f'(g(a))$ this is since f is differentiable at $g(a)$ but by our definition of ϕ $f'(g(a))$ is nothing but $\phi(g(a))$

so the limit of ϕ of y as y approaches $g(a)$ is same as $\phi(g(a))$ and this implies by definition of continuity that ϕ is continuous at $g(a)$

so this is one important fact now note that $f(y) - f(g(a))$ this is equal to $\phi(y) \cdot (y - g(a))$ this is true for all y

so if y is not equal to $g(a)$ then we know that $\phi(y) = \frac{f(y) - f(g(a))}{y - g(a)}$ and therefore by multiplying by $y - g(a)$ $f(y) - f(g(a))$ is equal to $\phi(y) \cdot (y - g(a))$

so of course this is true if y is not equal to $g(a)$ but if y is equal to $g(a)$ then you notice that left hand side is $f(g(a)) - f(g(a))$ which is zero and right hand side if you put y equal to $g(a)$ is again zero

so the above equality is true even if y is equal to $g(a)$

so now to find the limit of $f(g(a) + h) - f(g(a))$ we just put

so put y is equal to $g(a) + h$ then what do we get $f(g(a) + h) - f(g(a))$ is equal to $\phi(g(a) + h) \cdot (g(a) + h - g(a))$

so therefore if h is non zero we can divide write $f(g(a) + h) - f(g(a))$ divided by h this is equal to $\phi(g(a) + h) \cdot \frac{g(a) + h - g(a)}{h}$ now in the right hand side

so in r h s $\lim_{h \rightarrow 0} \frac{f(g(a) + h) - f(g(a))}{h}$ this we know is equal to $g'(a)$ because g is differentiable at a and what about the other limit we get $\lim_{h \rightarrow 0} \phi(g(a) + h)$ but what we have seen is that ϕ is continuous at $g(a)$

so this is equal to $\phi(g(a))$ since ϕ is continuous at $g(a)$ but what is $\phi(g(a))$ by definition of the function ϕ this is equal to $f'(g(a))$

so therefore $\lim_{h \rightarrow 0} \frac{f(g(a) + h) - f(g(a))}{h}$ is equal to $f'(g(a)) \cdot g'(a)$ that is $f \circ g'$ of a is equal to f' of g of a times g' of a

so this proves the chain rule

so one comment

so if you see when we wrote this difference coefficient as $\frac{f(g(a) + h) - f(g(a))}{g(a) + h - g(a)}$ here we have to assume that this $g(a) + h$ is not equal to $g(a)$ for small enough h

so if you see this might happen

so remark $g(a) + h - g(a)$ is not equal to zero might not hold true for small enough h no matter how small h you choose

so so as an example consider the function g of x is equal to $x^2 \sin \frac{1}{x}$

so this function first check that for this function g is differentiable at zero
so let us check that claim g is differentiable at 0 and $g'(0)$ is equal to 0
so let us calculate

so limit of h going to 0 of $g(0+h) - g(0)$ by h this is equal to
limit h going to 0 $g(h) - g(0)$ is $h^2 \sin \frac{1}{h}$ let me write this is defined
for x not equal to 0 and 0 if x is equal to 0 .

so g of x i have to define at 0 also and then this function we claim is
differentiable at zero

so this is $h^2 \sin \frac{1}{h} - g(0)$ is zero divided by h this is
same thing as limit h going to 0 of $h \sin \frac{1}{h}$ now we have dealt with
limits of this form

so this is equal to zero this is because we know that $\sin \frac{1}{h}$ is always
less than equal to one

so this implies that $h \sin \frac{1}{h}$ is less than equal to h and
of course this is greater than equal to zero and then by sandwich theorem limit
of $h \sin \frac{1}{h}$ is equal to zero

so this function $x^2 \sin \frac{1}{x}$ for x not equal to 0 and 0 at x equal to
 0 this function is differentiable at 0 and the derivative is equal to 0 but if
you see for this function but g of if i take any $\frac{1}{m\pi}$ this is equal to 0
for all integers m this is because $\sin \frac{1}{m\pi}$ is zero for all m
in integers

so g of $\frac{1}{m\pi}$ if you do it will be equal to $\frac{1}{m^2\pi^2} \sin m\pi$ which is 0
and notice that if you take any interval around 0

so no matter what interval you take there exist for large enough m $\frac{1}{m\pi}$
will be in this

so for any δ greater than zero $\frac{1}{m\pi}$ this belongs to $(-\delta, \delta)$
for some m in integers

so this is because when you you might think that to prove this it is enough to
do this for when $g(a+h) \neq g(a)$ and think that if the
function is not constant g is not constant then it must happen that in small
enough interval $g(a+h) \neq g(a)$ but that is not true this
example shows that now let us write this chain rule in some other forms

so so if we write y is equal to let me write y equal to $g(x)$ and u is equal
to $f(y)$ right

so this is writing the composite function

so that is u is nothing but $f(g(x))$ which is $f \circ g$ of x right

so any composite function composition can be written like this you define y to
be equal to $g(x)$ the inner function and then u is equal to $f(y)$

so then the chain rule says that if g is differentiable at x and f is
differentiable at $g(x)$ then $(f \circ g)'(x)$ is equal to $f'(g(x)) \cdot g'(x)$

so what is $g'(x)$

so that is we can write $f \circ g$ is nothing but the function u

so this is same thing as $\frac{du}{dx} = f'(g(x)) \cdot g'(x)$

so this is $f'(y)$ which is $\frac{du}{dy} \cdot \frac{dy}{dx}$ right we have y equal
to $g(x)$ this implies that $\frac{dy}{dx} = g'(x)$ and u is $f(y)$

so $\frac{du}{dy}$ is nothing but $f'(y)$ which is same as $f'(g(x))$

so we can remember the chain rule as if i have u is equal to $f(y)$ and y is g
of x then to find the derivative $\frac{du}{dx}$ you first find the derivative of u with
respect to y and then multiply it by derivative of y with respect to x

so this is easy to remember because if you see if this was usual division then

this dy cancels and then we get $du dx$

so that is the way you can remember it but note that $du dy$ this is just the symbol for the derivative it is not the quotient of two things

so now let us see some examples let f of x is equal to say x square plus one cube and then find f' of x right

so what are the ways you can do

so one way is that you can expand x square plus one cube this will be equal to x to the six plus three times x to the four plus three x square plus one and therefore now we know the derivative of x to the n therefore f' of x is equal to $6x$ to the 5 plus $12x$ cube plus $6x$

so this this is one way another way is you could use product rule

so if i want to find the derivative we write x square plus 1 times x square plus 1 square and then this is our f of x

so f' of x is the derivative of the first function $2x$ times x square plus 1 square plus x square plus 1 times the derivative of x square plus one square and for this you again use product rule

so this is $2x$ times x square plus one plus x square plus one times $2x$ which is equal to $4x$ times x square plus one and therefore f' of x is $2x$ times x square plus one square plus $4x$ times x square plus one square this is $6x$ times x square plus one square and you can see that these are this is same as the previous answer this is same as $6x$ times x to the four plus $2x$ square plus one and if you multiply we get $6x$ to the five plus $12x$ cube plus $6x$

so we get the same answer but there is another way to do it using chain rule

so using chain rule

so we have f of x is x square plus one cubed

so this is equal to g of x cube where g of x is x square plus 1 right

so now f of x is g of x cube where g of x is x square plus one by chain rule the derivative is f' of x is equal to the derivative of q will give me 3 times g of x square times g' of x this is equal to 3 times g of x is x square plus 1 square times g' of x gives me $2x$

so this is equal to $6x$ times x square plus 1 square which is the same answer that we got using the product rule or you can write ah y is equal to x square plus one cube this you write as u cubed where u is x square plus 1 and then we know that $dy dx$ y is a function of u now

so they write $dy du$ times u is a function of x

so du by dx and $du dy du$ is the derivative of u cube gives me three u square times $du dx$ gives $2x$ and then you have to write everything in terms of x

so this is three times x square plus one square times $2x$ right

so using chain rule it makes the calculation easier and we will see some more examples

so find said the derivative of the sine of x square

so here if you see we cannot even use product rule or do any simplification to find the derivative

so either you have to find using the limit or if you notice that you can use chain rule here

so so we write this

so write y is equal to sine of u and u is equal to x square

so $dy du$ will give $\cos u$ and $du dx$ is $2x$

so the derivative $dy dx$ this by chain rule is $dy du$ times $du dx$ which is equal to \cos of u which is x square times $2x$ right

so what this chain rule does is that you have sine of something you know the derivative of sine is cosine

so you find the derivative of the outer function first and then evaluate it at this inner function and then you find the derivative of the inner function

so after some practice you will be able to write this directly let us make it more complicated and suppose i want to find $\frac{d}{dx}$ of sine square of say x^3

so here if you notice sine square x^3 this can be written as composition of three functions

so sine squared x^3 this is equal to sine of x^3 and then you square this right

so here the outer most function is you square this and then you have sine of x^3

so the derivative this is my function y

so $\frac{dy}{dx}$ first you have this u^2 the derivative of that will give me 2 times sine of x^3 and then we have to write derivative of sine x^3 now like the previous example this derivative sine x^3 is nothing but first you take the derivative of sine you get cosine of x^3 and then derivative of x^3 will give three x^2

so this gives me the derivative let me do maybe one more example

so i can make it some sign of sine of cosine of x

so if you have to find the derivative $\frac{dy}{dx}$ here the outer most function is sine of something

so you get cosine of this whole thing then you have to take the derivative of sine of this

so times ok let me write once more $\frac{d}{dx}$ of sine of cosine x^3 plus x then you again use chain rule

so this will give me cosine of cosine x^3 plus x and then the derivative of cosine x^3 plus x again you use chain rule for that

so that gives me negative sign x^3 plus x and then multiplied by derivative of the inner most function x^3 plus x will give $3x^2$ plus 1 right

so using chain rule we see that we can easily find the derivative if we have composition of more than two functions then you use chain rule repeatedly to find the derivative of the composition now next thing we will like to see if we can find derivatives of inverse functions

so for example you would have studied about inverse trigonometric functions

so we would like to ask what is $\frac{d}{dx}$ of sine inverse x $\frac{d}{dx}$ of cosine inverse x $\tan^{-1} x$ and

so on

so so suppose let y equal to f of x and suppose this function

so suppose f of x has an inverse let me write g of x

so what does that mean that is f of g of x is equal to g of f of x and it is equal to x right

so the inverse means that if you have you take f of f^{-1} of x you get x and f^{-1} of f of x is x

so if we have y equal to f of x then x can be written as f^{-1} of y which is same as here g of y

so now to find the derivative

so if we look at therefore g' of y this is nothing but $\frac{dx}{dy}$ right and what we know is that since f of g of x this is equal to x which is this implies g of x we are writing

so we have if i take the derivative of this then with respect to x i get 1 is equal to $\frac{d}{dx}$ of f of g of x and this by chain rule is equal to f' of g of x times g' of x what we would want is to find the derivative g' of x

so if f' of g of x is not equal to zero then g' of x is $\frac{1}{f'$ of g of $x}$ right

so what we get is that

so the derivative of if f^{-1} denotes the inverse of f then $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

so what we need is that $f'(f^{-1}(x))$ should be non zero

so for example let us try to calculate the derivative $\frac{d}{dx}$ of sine inverse of x

so by this formula this is equal to one by the derivative of sign

so let me write one by $f'(f^{-1}(x))$ where $f(x)$ is equal to sine of x

so the derivative of sine of x gives cosine

so this will be cosine of sine inverse x and what is cosine of sine inverse x

so if y is equal to sine inverse x this means that x is same as sine of y and we want to find what is cosine of y this implies cosine of y cosine square y is equal to one minus sine square y which is one minus x square

so cosine of y is equal to plus or minus square root of one minus x square now what do we know about sine inverse x this is defined for x between minus one and one right because sine of x range is between minus one and one

so sine inverse x is defined for x in minus one one and this sine inverse x this belongs to 0 to $\frac{\pi}{2}$ if x is in 0 to 1 and sine inverse x is in minus $\frac{\pi}{2}$ to 0 if x belongs to minus 1 to 0 .

so if x is positive then sine inverse x is in the first quadrant between 0 and $\frac{\pi}{2}$ and if x is negative then sine inverse x is in minus $\frac{\pi}{2}$ to $\frac{\pi}{2}$ right.

now what about cos of this but cos of θ is positive if θ is between minus $\frac{\pi}{2}$ to 0 minus $\frac{\pi}{2}$ to $\frac{\pi}{2}$ right in the first quadrant and fourth quadrant would cosine is an even function

so it is always positive in minus $\frac{\pi}{2}$ to $\frac{\pi}{2}$

so therefore cos of sine inverse of x this is positive for all x between minus one and one

so this cos of sine inverse x we wrote here cos of sine inverse x is plus or minus square root of one minus x square but we know that it has to be always non negative

so cos of sine inverse x is equal to square root of one minus x square this is true for all x and it is true for in the closed interval minus 1 to 1 but then

so therefore the derivative $\frac{d}{dx}$ of sine inverse x this is equal to $\frac{1}{\sqrt{1-x^2}}$

so i will do one more the derivative of tan inverse x

so if you write y equal to tan of x

so tan inverse x and we want to find $\frac{dy}{dx}$

so sometime it is easier to write this is same thing as writing x equal to tan of y and therefore if i write $\frac{dx}{dy}$ this is equal to derivative of tan y gives me secant square y

so therefore $\frac{dy}{dx}$ note that by chain rule by chain rule this is equal to one by $\frac{dx}{dy}$ and this is equal to $\frac{1}{\sec^2 y}$ secant square y is same as $1 + \tan^2 y$ and $\tan y$ is equal to x

so this is $\frac{1}{1+x^2}$

so therefore the derivative of tan inverse x is equal to one by one plus x square right you should try to calculate you know the derivative of other inverse trigonometric functions and then in the next class i will write down the formulas for those and then we will also find derivatives of some other functions thank you