welcome to the second lecture on derivatives

so in the last lecture we discussed the concepts of continuity of a function at a point and also we discussed what do we mean by derivative of the function at a point and then we saw some properties of derivatives today we will first see the relation between continuity and differentiability and then we will calculate derivative of some more functions

so the first thing i am going to discuss is is there any relation between continuity and differentiability

so first of all lets see an example

so consider the function f of x equal to mod x for x in r

so let us try to draw the graph of this function

so f x equal to mod x for any x positive this is equal to x and for x negative it is equal to minus x

so this is also equal to f x is equal to x for x greater than equal to zero and minus x for x less than zero right this is very simple but useful representation of this function and the graph looks like this

so this is f x equal to mod of x now let us ask if the function is

so is f x continuous at x equal to zero

so here it was easy to draw the graph of this function and from this graph one can see that the function is continuous at zero but we can also calculate the limit

so the limit of f of x

so to calculate the limit here it is useful to calculate the left hand and right hand limit

so if you calculate the right hand limit of f of x

so f of zero plus h as h goes to zero what is this this is equal to limit h going to zero plus of f of h is mod h but because we are taking s to be positive this is same thing as limit of h as h goes to zero plus which is equal to zero similarly the left hand limit limit h going to 0 minus of f of 0 plus h this is equal to limit of h going to zero minus of mod h but here because h is going to zero minus mod h is equal to minus h but h is going to zero

so minus zero is also equal to zero

so limit exist

so therefore limit of f of x as x approaches zero this is equal to 0 also f of 0 is mod 0 which is 0

so therefore f of 0 is the limit of f of x as x approaches zero hence f x is continuous at x equal to zero now what about differentiability is f x differentiable at x equal to zero

so we have to ask this limit does limit of f of 0 plus h minus f of 0 by h as h goes to 0 exists lets look at what is f of zero plus h minus f of zero by h if we take any h non zero and look at this difference cosine this is equal to f of h minus f of 0 by h and f of h is mod h f of 0 is 0 divided by h

so we get mod h by h now we know that mod h is equal to h if h is positive

so this is equal to one if h is positive and if h is negative then mod h is equal to minus h

so minus h by h will give minus one

so we see that this say difference cosine is equal to constant 1 if h is positive and it is equal to minus 1 if h is negative

so therefore the left hand limit and right hand limit is not equal to right hand limit of f of zero plus h minus f of zero divided by h

so we conclude that hence this limit h going to zero of f of zero plus h minus f of zero divided by h this does not exist which means that the function f of x is not differentiable at x equal to zero

so what is the conclusion f of x continuous at x equal to a does not imply

applied at f of x is differentiable at x equal to a let me come back to this graph and also explain you geometrically how you can deduce that the function is not differentiable at this point zero

 so we have seen the geometric interpretation that the derivative if it exists it is equal to the slope of the tangent line at that point now if you look at the graph of this function there is no unique tangent line because for any positive thing you see this is the tangent line but if x is any in negative then we have this line

 so there is no unique tangent line here and in fact you see that the right hand derivative is nothing but slope of this line which is 1 and left hand derivative is slope of this line which is minus 1 and they are not equal

 so the function is not differentiable in general whenever in the graph of your function if you see a corner point then the function will not be differentiable at that point right

 so continuity we said that that is graphically it means that you can draw the graph of the function without lifting your pen and differentiability means that the function should not have any corner though for complicated functions we cannot just use this interpretation

 so we have to know the regress definition also

 so now what about the converse

 so the converse is true

 so the theorem is that if f of x is differentiable at some point x equal to a then f of x must be continuous at x equal to a and the proof is very simple you have to just note this

 so since f of x is differentiable at x equal to a we have f prime a which is equal to limit of f of a plus h minus f of a divided by h this exists right now to check the continuity of the function we need to look at the limit of f of x as x approaches a

 so to check the continuity at x equal to a we need f of a to be equal to limit of f of x where x approaches a now note that this limit we can also write as limit of f of a plus h where h approaches zero right by putting x equal to a plus h if x approaches a then h which is equal to

 so this is by put x equal to a plus h that is h is x minus a and then as x approaches a h approaches 0.

 so we have to check whether this limit f of a plus h as h approaches 0 exists or not and whether it is equal to f of a now if you see this is same thing as

 so limit f of a plus h h approaching 0 equal to f of a this is equivalent to writing limit h going to 0 f of a plus h minus f of a this is equal to 0 right because the limit of this constant function f of a as h approaches 0 that is f of a

 so we have this now this is the numerator of the difference quotient in the definition of derivative

 so but for any h naught zero f of a plus h minus f of a this can be written as h times f of a plus h minus f of a divided by h right i just multiplied it multiplied and divided by h to get this and what we know is that this limit exists

 so since limit of h as h goes to 0 this is equal to 0 and limit h going to zero f of a plus h minus f of a by h this also exists by the product rule for limits we have limit of h going to 0 f of a plus h minus f of a if limit of both the functions exist then the limit of the product is product of the limit

 so that is equal to 0 times f prime a which is equal to 0 hence f x is continuous at x equal to a

 so what we have seen is that differentiability of a function at a point implies

continuity of the function at that point whereas the continuity need not imply differentiability that we have seen by a counter example

so next to calculate some more derivatives let's learn what is called product rule for derivatives

so what this says is that if i take two functions and look at the product

so d by d x of f x times g x this is equal to f prime x times g x plus f x times g prime x provided f prime x and g prime x exists

so we will prove this

so let let us write u of x equal to f x times g x then we have to look at

so for any h non zero u of x plus h minus u of x divided by h you have to look at what is this

so let me write this this is equal to u is the product of f and g

so i get f of x plus h times g of x plus h minus f of x times g of x divided by h now we do just a little algebraic manipulation here

so we add and subtract we write this as f of x plus h g of x plus h minus f of x times g of x plus h and then we again add this same quantity f of x times g of x plus h and then we have minus f x g x divided by h

so we write like this and then group the first two terms and the last two terms now if you see in the first two terms we have g of x plus h is common

so we got u of x plus h minus u x by h this is equal to f of x plus h minus f of x divided by h this times g of x plus h and then plus in the next two terms we have f x is common

so i'll write f x times g of x plus h minus g of x divided by h now if we look at the two terms on the right hand side now limit as h goes to zero of f of x plus h minus f of x divided by h this we know is equal to f prime x because f is differentiable at x what about the second term in the this product limit of h going to 0 of g of x plus h this is simply equal to g of x why this is because g is continuous at x since it is given to be differentiable at x right

so note that we are using the theorem that we proved that if g is differentiable at x then it is also continuous

so the limit of this because as h goes to 0 x plus h approaches x and because it is continuous at x this limit must be equal to the value of the function at x similarly the other term we have limit of h going to 0 g of x plus h minus g of x divided by h this is equal to g prime of x and the first term is independent of h this is f of x

so that limit is f of x

so therefore we have that this limit exists u prime x which is equal to the limit of h going to zero u x plus h minus u x divided by h this is equal to the first limit here is f prime x the limit of the second this is g of x plus here the first term is simply f of x times the limit of the second one is g prime x right

so this is very important formula and let me derive this formula let me write as a warning that d by d x of f x times g x you should not write that this is simply equal to let me write f prime x times g prime x this is not true right for example if you take f x equal to g x equal to x then f prime x is one which is equal to g prime x also but what about f x times d x but f x times g x is equal to x square

so d by d x of f x g x is d by d x of x square which we have seen is equal to 2 x this is not equal to f prime x times g prime x

so in the beginning many students make this mistake and when they see product of functions and then limit a derivative they write as product of the derivative which is absolutely wrong

so so now using this we can derive some more derivatives

so so example suppose you want f x equal to say x cube and then find f prime x

so what we know is that we know derivative of x square and x

so x cube f of x here is equal to x square times x therefore f prime x will be equal to d by d x of x square times the second function is x plus x square times d by d x of x which is equal to the first derivative is 2 x times x plus x square times 1

so we get two x square plus x square which is three x square

so let's calculate the derivative of to calculate d by d x of x to the n where n is any natural number

so so one one way would be to use the product rule rapid repeatedly to get this derivative or you can try to calculate the limit directly also

so let us look at

so we have f x equal to x to the n

so for h non zero if i look at f of x plus h minus f of x divided by h this is equal to x plus h to the power n minus x to the n divided by h and if you seen the binomial theorem then x plus s to the n we can write as x to the n plus n choose one x to the n minus one times h plus n choose two x to the n minus two h square and

so on till the last term is h to the n and then minus x to the n divided by h now here if you note this x to the n cancels with this and now you note that every term has h in common

so we get this is equal to h times n choose one is simply n x to the n minus one plus we have n choose two x to the n minus two times h plus other terms which contains h h to the n minus 1 divided by h

so by cancelling this h we see that every term except this first term contains h

so this approaches n times x to the n minus 1 as h goes to 0 because all the other terms we get as h or h square

so they go to zero

so this proves that

so therefore d by d x of x to the n is equal to n times x to the n minus 1 for every natural number n later we will see that this is in fact true even if n is not a natural number

so first thing that now from this you can easily see calculate d by d x of x square if you have to calculate this is two times x to the two minus one

so that is two x d by d x of x cube is three times x to the three minus one which is three x square d by d x of x to the four

so you bring the exponent down and then reduce the exponent by 1 this is 4 x cube and

so on

so remark the above formula for derivative of x to the n actually holds true for any real number n this will be proved later but let us try to calculate x to the n for some negative

so derive the derivative of we take f of x is equal to 1 by x then what is f prime x

so let us do by using our first principle that is definition of the derivative

so if we look at f of x plus h minus f of x divided by h this is equal to one by x plus h minus one by x divided by h and then if you simplify this you get h times x times x plus h and then numerator is x minus x plus h

so here x cancels and we get minus h y h x x plus h and then you can cancel this h and this is equal to minus one by x times x plus h this is true for any h non zero and therefore the limit as h goes to zero of f of x plus h minus f of x by h this is equal to minus 1 by x square because here in the denominator you see x plus h approaches x plus 0

so x times x gives x square this is true for for all x not equal to zero right

so this function is not even defined at zero

so we cannot talk about the derivative at zero but for any x not equal to zero we get that d by d x of 1 by x is equal to minus 1 by x square for x not equal to 0.

note that this i can write that this is this same formula for let me write this agrees with the formula for d by d x of x to the n equal to n x to the n minus one of for n equal to minus one this will give this gives d by d x of x to the minus one is equal to n is minus one here times x to the minus one minus one which is equal to minus x to the minus two which is same as minus one by x square right and if you want for other negative exponents then you can may use product rule

so can use product rule to calculate d by d x of x to the minus two or d by d x of x to the minus three etcetera

so for example d by d x of x to the minus two is same as d by d x of one by x times 1 by x and now you use the product rule this is same thing as minus 1 by x square this is the derivative of 1 by x times the second function is 1 by x plus the first function 1 by x times the derivative of the second function minus 1 by x square this is by product rule and this gives minus 1 by x cube minus 1 by x cube

so minus 2 by x cube of course this again agrees with the formula the derivative of x to the n is n times x to the n minus 1 ok now seeing this we can derive another formula for we did for product we can also define derive the formula for the cosine of two functions

so let's try to calculate what is say

so suppose f of x is differentiable differentiable at x equal to a and

so then we ask for what about let g of x equal to one by f of x just like we calculated the derivative of one by x lets try to see if we can calculate the derivative of one by f x

so if you look i will write what the derivative is that we will get from the proof

so if i look at g of a plus h minus g of a divided by h this is equal to one by f of a plus h minus one by f of a divided by h and this is equal to f of a minus f of a plus h divided by h times f of a f of a plus h and this is same thing as negative of f of a plus h minus f of a divided by h lets pull this out and then times 1 by f of a times f of a plus h now lets see if the limit exists now what we know is that this limit f of a plus h minus f of a by h this approaches f prime a and here i have 1 by f a times f of a plus h

so this f of a plus h this approaches f of a

so what we get is that therefore the limit g prime a is equal to minus f prime a divided by f of a squared of course here when i am writing g of x equal to 1 by f x and we are asking for whether this is differentiable at x equal to a

so g of a must be defined

so we must have that and f of a is not equal to 0 then g prime a is equal to minus f prime a divided by f of a squared and then we can derive more general quotient rule

so this says that if i have f x and g x which are both differentiable at some point and g of a is not equal to 0 then the derivative of the cosine d by dx of fx by gx this is nothing but the derivative f prime x times g of x minus f of x times g prime x divided by g of x squared and the proof you can do it by just writing the ah as limit of the difference cosine but here note that we have derived the product rule and derived the derivative of reciprocal of function

so we can use that

so d by d x of f x by g x f x by g x i can write as the product f x times one by g x and then the first one by product rule this is equal to f prime x times d

by dx of let me write it here

so this is the derivative of the first function times one by g x plus f of x times the derivative of the second function d by d x of one by g x this is by the product rule and then we know what is the derivative d by d x of one by g x

so this is equal to f prime x over g of x plus the second one is plus f x times the derivative is given by minus g prime x divided by g of x squared right and then if you take the common denominator g of x square we get f prime x times g of x minus f x times g prime x right

so let's summarize the product rule and quotient rule

so the product rule sometimes we will also write using this notation u v if these are two functions then the derivative of u v is u prime times v plus u v prime and the quotient rule is if i have u by v the derivative prime is equal to u prime v minus u v prime divided by v square this is product rule and this is the quotient rule and these rules are very important to calculate derivatives

so ok lets calculate some more derivatives

so one i will do is another example derivative of say f x equal to square root x this is defined for all x greater than zero

so we want to calculate the derivative of this function at any positive x

so f of x plus h minus f of x by h if x is any positive real number this we can write as square root of x plus h minus square root of x divided by h and if x is positive then for small h x plus h is also positive

so we can talk about this square root and then we want to find the limit of this as h goes to zero

so while calculating limit we have calculated limits of this type

so one way to do this is you multiply and divide by square root x plus h plus square root x by square root x plus h plus square root x and then what you get in the numerator you get x plus h minus x divided by h times square root x plus h plus square root x and then in the numerator x cancels and then you can cancel the h you get one by square root x plus h plus square root x which approaches one by two square root x as h goes to zero

so what we got is that d by d x of square root x is equal to one by two square root x for all x greater than zero again note that this agrees with the formula x to the n because if i write if we write square root x as x to the power one half then the derivative d by d x of square root x which is equal to 1 by 2 square root x which is nothing but 1 by 2 times x to the power negative half which is same as 1 by 2 times x to the power half minus one

so this also agrees with with the formula d by d x of x to the n is equal to n times x to the n minus one though we derived this formula only for positive integers but we have seen that it agrees for n equal to minus one it agrees for n equal to square n equal to half and later we will see that this is in fact true for any n another example i will do is

so far we have only calculated derivatives of some powers of x

so lets calculate derivative of f x equal to sin x

so if we look at f of x plus h minus f of x divided by h this is equal to sine x plus h minus sin x divided by h and then you recall what is the formula for sine of c minus sine of d this is equal to two times cos of plus d by two okay sin c minus sin d is equal to two times cosine c plus d by two times sine c minus d by two

so we have sine x plus h minus sin x this will be equal to two cos c plus d is two x plus h by two sine h by two and therefore f x plus h minus f x by h this is equal to two cos two x plus h by two sine h by two divided by h and then we have to ask whether this limit exists

so this is equal to cos of x plus h by two times sine h by two divided by h by two and now remember that the limit as h goes to zero of sin h by h is one

so therefore the limit of f x plus h minus f x by h and the first term cos x plus h by 2 this goes to cos of x times one this is because limit of sine h by h as h goes to zero this is equal to one

so therefore we got that d by d x of sine x is equal to cos x this is again this will be useful formula for you

so then one can ask for the derivative of cos x derivative of

so again if you calculate this derivative d by d x of cos x this will be equal to limit of h going to zero of cos of x plus h minus cos of x by h and again if you use the formula for cos c minus cos d and then you can show that this limit is equal to minus of sine x

so this i leave as an exercise i leave as an exercise for you to check that the derivative of cos x is minus sign x and now because we know product rule and quotient rule we can calculate derivatives of other trigonometric functions

so what is d by d x of tan x

so we know that tan x is nothing but sine x by cos x and then we use quotient rule

so this is equal to the derivative of sine x times cos x minus sine x times derivative of cos x divided by the denominator is cos x square and this is by the quotient rule now we have calculated derivative of sin x is cos x

so this is cos x times cos x and i have asked you to verify that derivative of cos x is minus sign x

so this is minus sin x divided by cos x square which we also write as cos square x

so we get in the numerator cos square x plus sine square x divided by cos square x but you know that cos square x plus sin square x is 1 by cos square x

so this is equal to secant square x

so what we get is derivative of tan x is secant square x and now other trigonometric functions also you can calculate because those are just the reciprocals of these functions

so if i write d by d x of secant x then secant x is nothing but one by cos x and then we know the derivative of the reciprocal is given by negative of derivative of cos x divided by cos x square right this is by quotient rule or by our special thing one by f x derivative and then derivative of cos x is minus sin x

so we get this is sine x divided by cos square x and this generally we write in this form

so this i can write as sine x by cos x times one by cos x which is equal to tan x times secant x

so we will remember this formula as d by d x of secant x is equal to secant x times tan x similarly you verify that d by d x of cosecant x this is equal to minus of cosecant x times cot x and d by d x of one more left is cot x this is equal to minus cosecant square x

so these two are again exercise for you

so i will stop here in the next class we will learn the chain rule for derivative which is very useful to calculate derivative of many more functions thank you you