

hello everyone

so today i will be starting two very important concepts in calculus which are continuity and differentiability of a function

so so far we have studied what do we mean by limits of a function at a point and we'll see that the limits play central role in defining the continuity and differentiability of a function of point and in subsequent lectures we will see many applications of these concepts

so let me begin with defining what do we mean by a continuous function at a point

so i will discuss what do we mean by continuity of a function at a point

so we will always assume let f be a real valued function defined on a domain d

so suppose the function is defined on a domain d which is a subset of \mathbb{R} real number and suppose a belongs to d that is the function f is defined at f of a is defined

so we will say we say that f is continuous at a if f of a is equal to the limit of the function f of x as x approaches a

so recall that recall that for limit f of x x going to a to exist the function f of x must be defined in some interval some open interval containing a

so we say that f is continuous at a if

so f is f of x is continuous at x equal to a if the two condition first the limit of f of x as x approaches a exists

so first of all the limit of the function as x approaches a should exist and second f of a must be equal to the limit of f of x as x approaches a ah let me say here that for the limit to exist we want the function to be defined in some interval containing a but not necessarily but not necessarily at x equal to a right

so for the limit we do not require the function to be defined at x equal to a but the continuity definition is that the value of the function at a must be equal to the limit

so for f to be continuous for f to be f x to be continuous at x equal to a the function must be defined at x equal to a and f of a must equal the limit of f of x at x approaches a

so let us see some examples examples one

so consider f of x which is equal to 0 for x less than 0 and 1 for x bigger than equal to zero

so if you draw the graph of this function this function is zero for all x negative and for x greater than equal to zero f x is equal to one

so this is my x y and this is the graph of y equal to f of x

so here what we know is that here limit of f of x at x approaches 0 this does not exist this is because the left hand limit at zero of f of x this is equal to zero which is not equal to one which is the right hand limit at zero

so since the limit does not exist

so f of x is not continuous at x equal to zero even though f of x is defined at x equal to zero we have f of zero is equal to one but if you take any point other than zero but if a is not equal to zero then the limit of f of x as x approaches a this is equal to 1 if a is strictly positive and this is equal to zero if a is strictly negative right this you can see from this graph that if we take any a to be positive then the function is constant one in an interval containing this point a and therefore the left hand limit and right hand limit are both equal to one but if your a is somewhere here negative then in this interval the function is identically zero therefore the limit is zero also f of a is equal to 1 if a is positive and 0 if a is negative

so therefore the function f of x is continuous at all points except x equal to zero right also if x is not continuous at x equal to a we will say that f of x

is discontinuous at x equal to a

so intuitively what does continuity mean intuitive meaning of continuity

so if we can draw the graph of the function and suppose we have this point a

so this is the point a comma f of a

so if we can draw the graph of the function y equal to f of x then the continuity of f of x at x equal to a means that the graph is not broken near the point a comma f of a on the graph

so which means that you can draw the graph of this function near this point a comma f of a without having to lift your pen

so but we will see that this intuitive definition is not enough to make regressions of continuity

so for example

so for the previous example f x equal to 1 for x greater than 0 and 0 for x less than zero or we said for greater than equal to zero it is one if you draw the graph near this point zero comma one you see for x greater than zero this is one but for x less than 0 it is 0 .

so you cannot have this not broken for this function

so here you see that f the graph effects has broken graph near zero comma one

so this example

so the previous example was where the function was not continuous or discontinuous at a point because the limit did not exist at that point but the next example lets consider the function f of x which is equal to 1 if x is not equal to 0 and 0 at x equal to zero

so this function it is one at all point except zero

so we draw a open circle here and this is the value one and at x equal to zero this is zero

so this is the graph of the function f of x

so in this case again by using the intuitive meaning you can see that this graph is broken at the point this point zero comma one

so the function is not continuous at x equal to zero in this case if you see here limit of f of x as x approaches 0 this exists and is equal to 1 but the value of the function at 0 this is given to be equal to 0 which is not equal to the limit of f of x as x approaches zero hence the function f of x is discontinuous at x equal to zero

so we have seen two examples in one the limit of the function did not exist at a point and therefore the function cannot be continuous at that point another example was where the limit exists the function is defined at that point but the value of the function is not equal to the limit

so therefore again the function is discontinuous at that point now let's see let's consider this example f of x which is equal to x times sine of 1 by x and because i am writing 1 by x this does not make sense for x equal to 0

so i defined this to be the value of the function for x not equal to zero and this is not defined for x equal to zero

so for x equal to 0 i will define the value of the function to be 0 .

so this is the function which is 0 at 0 and for any non-zero x the function is x times sine 1 by x

so here what can you say is f continuous at x equal to zero

so in this case if you see if you try to draw the graph of the function x times sine one by x it is very difficult to draw it near the point 0 because as you go close to 0 this function sine 1 by x it keeps oscillating

so here let me say it is difficult to draw the graph

so it is difficult to see whether the graph is broken or not but let us try to

calculate the limit if we can

so but can we calculate the limit of the limit x going to 0 f of x can be calculated as follows i think we have seen this example maybe before also

so how do i

so that the limit exists

so if you look at f of x lets look at what is \sin of x this is equal to \sin of x times \sin of 1 by x which is less than equal to \sin of x times \sin of 1 by x and since \sin of 1 by x we know that it is always between minus one and one for all x not equal to zero of course for x equal to zero \sin of 1 by x is not defined but for any x not equal to zero we know that \sin of y is always between minus one and one therefore \sin of 1 by x this is less than equal to one therefore \sin of x times \sin of 1 by x this is less than equal to \sin of x and of course this is because we are taking the absolute value \sin of x is greater than equal to 0 .

now we have seen the sandwich theorem

so since limit of x going to 0 of x is 0 which is also equal to the limit of the left hand side 0 as x going to 0 by the sandwich theorem the limit of $x \sin$ of 1 by x we write \sin of 1 by x as x goes to 0 this is equal to 0 and this implies limit of $x \sin$ of 1 by x is also zero

so the limit of the function therefore we have that f of 0 is defined to be equal to 0 and this is equal to the limit of f of x as x approaches 0 therefore by the definition of continuity

so f of x is continuous at x equal to zero

so this example is important because here drawing the graph of the function and then inferring whether the function is continuous at x equal to 0 or not that is difficult but by calculating the limit of the function at 0 we saw that that is equal to the value of the function f of 0 and hence the function is continuous at zero

so later we will let's see some more examples of some functions where it might not be very easy to deduce whether the function is continuous at a point or not but let me go ahead and discuss another concept which is differentiability of a function

so let f of x be a real valued function which means that the range of the function is a subset of real number let f of x be a real valued function when we say that f of x is differentiable at x equal to a if the limit limit of h going to zero of f of $a + h$ minus f of a divided by h this exists

so note that for f of x to be differentiable at x equal to a the function f of x must be defined in an open interval containing x equal to a this is because see we are saying that this limit of f of x plus f of $a + h$ minus f of a divided by h this limit should exist

so first of all you see that here there is f of a

so we must have that f should be defined at x equal to a also we are writing f of $a + h$ and then taking the limit as h going to zero

so that means that h can be positive or negative and small the real number

so the function must be defined in some interval

so if a is here then in some interval near this the function must be defined then only we can talk about the differentiability of the function

so now let me explain what this ratio that i wrote f of $a + h$ minus f of a by h is

so if you look at the graph of the function let me write geometric interpretation

so let me draw a function and let me take this point to be x equal to a

so this is f of a now i have this point which is let me call this point as p which is whose coordinates are a comma f of a and lets look at another point

which is $a + h$ here i am taking h positive then there is another point q here whose coordinates are $a + h$ and $f(a + h)$

so we have this is $f(a + h)$ now if you look at if i draw this line segment joining the point p and q and i draw a horizontal line here then this part is h and this vertical part this is $f(a + h) - f(a)$

so this ratio the ratio $f(a + h) - f(a)$ by h this is nothing but is the slope of the line joining the points p which is given by $a, f(a)$ and q which is the point $a + h, f(a + h)$ now what happens as h approaches 0

so we have to ask whether this limit of this function this ratio as h approaches 0 exists or not

so if you see from this graph as h approaches 0 this point q approaches p as h tends to 0 the point q tends to p and what happens to and the secant that is the line segment joining these points on the curve is the line segment joining p and q approaches the tangent line at at the point $a, f(a)$

so maybe let me draw another diagram we have this point p here and there is this tangent line

so as this q moves and approaches this point p the secant line approaches this tangent line and the slope of the secant line approaches the tangent line

so limit $f(a + h) - f(a)$ divided by h this is the slope of the tangent line provided the limit exists okay

so if the function is differentiable

so if $f(x)$ is differentiable at x equal to a then we denote by $f'(a)$ this limit $f(a + h) - f(a)$ divided by h right

so if the function is differentiable that is this limit exists then this limit is defined to be $f'(a)$ and $f'(a)$ is called the derivative of $f(x)$ at x equal to a right

so the derivative is the limit of this difference cosine if this limit exists and if the limit does not exist then we will say that the function is not differentiable if the limit of the difference cosine which is $f(a + h) - f(a)$ by h does not exist then we say that $f(x)$ is not differentiable at x equal to a now let us look at some examples

so examples

so first one the simplest function is let us take f from r to r where $f(x)$ is equal to c a constant

so let us look at the constant function $f(x) = c$ for all x in r now here what is $f'(a)$

so for any a in r $f(a + h) - f(a)$ this is equal to $c - c$ which is 0 for all h therefore the limit of $f(a + h) - f(a)$ by h this is equal to zero because the numerator is zero

so therefore f is differentiable at every a and $f'(a)$ is equal to zero

so for the constant function the derivative is 0 everywhere next example let us look at $f(x) = x$ now let us try to calculate again the limit

so $f'(a)$ this is equal to limit of h going to zero $f(a + h) - f(a)$ divided by h which is equal to limit of h going to zero $f(a + h) - f(a)$ is $a + h - a$ divided by h here $a - a$ cancels

so this is equal to limit h going to zero of h over h and for any h non zero h over h is one

so this limit is equal to one

so for $f(x) = x$ the derivative $f'(x)$ is equal to one for all x in r ok lets try to calculate one more lets look at $f(x) = x^2$ now if i look at $f(x + h) - f(x)$ divided by h if this limit as h approaches zero exists that will be the derivative $f'(x)$

so this is equal to $x + h$ square minus x square divided by h and then we write $x + h$ square as $x^2 + 2hx + h^2$ minus x^2

divided by h then x^2 and x^2 cancels

so this we are writing always for h not equal to zero and this is equal to h times $2x + h$ by h

so this is equal to $2x + h$ now we see that this goes to $2x$ as x goes to zero

so therefore $f'(x)$ which is the limit of h going to zero of $f(x+h)$ minus $f(x)$ over h is equal to $2x$

so let me introduce one more notation

so this $f'(x)$ is also denoted by $\frac{d}{dx}$ of $f(x)$ or $\frac{dy}{dx}$ where y is equal to $f(x)$

so what we have got

so far is

so we can write $\frac{d}{dx}$ of any constant function is zero $\frac{d}{dx}$ of the function $f(x) = x$ is equal to one and we saw that $\frac{d}{dx}$ of x^2 this is equal to $2x$

so far we have seen these three derivatives now one important property is that suppose $f(x)$ and $g(x)$ are two functions which are differentiable at $x = a$ then the function $f(x) + g(x)$ is also differentiable at $x = a$ and we can write the derivative $\frac{d}{dx}$

so we will write this at $x = a$ of $f(x) + g(x)$ this is equal to $\frac{d}{dx}$ of $f(x)$ at $x = a$ plus the derivative of $g(x)$ at $x = a$ that is this is same as $f'(a) + g'(a)$

so let's see proof

so let me write $u(x) = f(x) + g(x)$ then we want to show that u is differentiable at a

so if I write $\frac{u(a+h) - u(a)}{h}$ this is equal to $\frac{f(a+h) + g(a+h) - f(a) - g(a)}{h}$ that is $\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}$ and this can be written as sum of $\frac{f(a+h) - f(a)}{h}$ plus $\frac{g(a+h) - g(a)}{h}$

so for any $h \neq 0$ we have this and what we know is that the limit of the right hand side both the limits exist

so therefore limit of $\frac{u(a+h) - u(a)}{h}$ as h goes to zero this is equal to limit of $\frac{f(a+h) - f(a)}{h}$ plus limit of $\frac{g(a+h) - g(a)}{h}$ this is by the sum rule of limits we have seen for limits that if the limit of $f(x)$ and $g(x)$ exist at some point then the limit of the sum also exists and limit of the sum is sum of the limit

so that's what we are using and this is same thing as saying that $u'(a)$ exists and is equal to $f'(a) + g'(a)$ another important property is that if I write $u(x) = c \cdot f(x)$ for some constant c in real number and $f'(a)$ exists that is f is differentiable at a then $u'(a)$ exists and $u'(a)$ is equal to $c \cdot f'(a)$

so to prove this again we write $\frac{u(a+h) - u(a)}{h}$ this is equal to $\frac{c \cdot f(a+h) - c \cdot f(a)}{h}$ which is equal to $c \cdot \frac{f(a+h) - f(a)}{h}$ and we know that this limit is equal to

so I will simply write this tends to $c \cdot f'(a)$ as h tends to zero therefore $u'(a)$ is equal to $c \cdot f'(a)$

so using these two results we can say more generally that if $u(x) = c_1 \cdot f(x) + c_2 \cdot g(x)$ and $f'(a)$ and $g'(a)$ exist then $u'(a)$ is equal to $c_1 \cdot f'(a) + c_2 \cdot g'(a)$ this is simply combining the previous two properties

so we know that $u'(a)$ is equal to by the sum rule this is equal to $\frac{d}{dx}$ at $x = a$ of $c_1 \cdot f(x) + c_2 \cdot g(x)$ at $x = a$ and by the constant multiple this is equal to $c_1 \cdot f'(a) + c_2 \cdot g'(a)$

times g' by the previous two results ok

so so lets say use this result to calculate some derivatives

so example let $f(x)$ equal to $x^2 - 3x + 2$ calculate f' at three if it exists

so what we know is that we know the derivative of x^2 derivative of x and derivative of constant

so since $\frac{d}{dx} x^2$ is $2x$ $\frac{d}{dx} x$ is one and the derivative of the constant two is zero we have that the derivative of $x^2 - 3x + 2$ is equal to $2x - 3$ times the derivative of x is one and plus zero

so this is nothing but f' of x

so in fact for any x the derivative exists at x and is given by $2x - 3$ and therefore f' at three you just have to plug in x equal to three is two times three minus three which is equal to three right

so using this sum rule and constant multiple rule we can calculate the derivative of for a combination of these functions if i know the derivative of each of them

so we will stop here and in the next class we will learn some more properties and then calculate derivative of some more functions thank you