

welcome students to the sixth and last lecture on determinants as i said earlier in this lecture i shall talk about solution of systems of linear equations you know that an equation is said to be linear if the underlying variables have exponent one for example $x + by = c$ or say $2x + 3y = 5$ the underlying variables here are x and y and since all of them have exponent is equal to one these are linear equations when we talk about systems of linear equations we consider multiple equations with multiple variables and we like to solve for the variables such that the obtained values satisfy all the equations simultaneously in this class we shall deal with either two equations with two variables or three equations with three variables now consider an equation a system of equations such as $2x + 3y = 5$ and say $x + 2y = 3$ we know that by $x = 1$ and $y = 1$ satisfy both the equations simultaneously is it the case all the time no example consider $2x + 3y = 5$ and $4x + 6y = 15$ suppose you are given two such equations it is very clear there cannot exist any x and y which satisfy both of them simultaneously because if i multiply the left hand side of this equation 1 with 2 we get $4x + 6y$ but on the right hand side we get 15 therefore it cannot be satisfied on the other hand if we consider $2x + 3y = 5$ and $4x + 6y = 10$ then we face a different problem since multiplying this equation by 2 we get the second equation we can get multiple solutions for the above system of equations for example $x = 1$ $y = 1$ therefore $2x + 3y = 5$ $4x + 6y = 10$ both are satisfied $x = -5$ $y = 5$ therefore $2x + 3y = -10 + 15 = 5$ $4x + 6y = -20 + 30 = 10$ $x = 2$ $y = 1$ by 3 therefore $2x + 3y = 4 + 3 = 7$ $4x + 6y = 8 + 3 = 11$ in fact we can compute in finite number any solutions for such a system of equation therefore given a system of equations we can have three possible situations either we can have one solution or we can have multiple solution or we can have no solution question is how do we judge what is the situation for a given system of linear equations

so we apply matrix and matrix inverse based techniques as follows let the systems of equations be $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$ $a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$ or in other words we are looking at n equations in n unknowns namely x_1, x_2, \dots, x_n note that we can represent such a system of equations in a matrix form $Ax = b$ where A is $n \times n$ x is a vector of or a one dimensional matrix of n rows and one column of variables and b is the right hand side again n rows and one column or a one one up to a one n a two one up to two n a n 1 up to a n n times $x_1, x_2, \dots, x_n = b_1, b_2, \dots, b_n$ thus very neatly we can represent the entire system of linear equations in a matrix form case one suppose A is non-singular then what will happen we know that we can compute A^{-1} this we have seen in our last class therefore by pre multiplying A^{-1} into $Ax = b$ or $x = A^{-1}b$ thus if A is non-singular we get unique solution for the system of linear equations example consider $2x + 3y = 8$ $3x - y = 1$ therefore in matrix notations $\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ multiplied by $x, y = 8, 1$ now determinant of A which is this matrix is equal to determinant of $\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$ is equal to $2(-1) - 9 = -2 - 9 = -11$ not equal to zero therefore A^{-1}

exists how to compute a inverse we know a inverse is equal to adjoint of a divided by determinant of a now the matrix is $\begin{bmatrix} 2 & 3 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ therefore adjoint of a is equal to we know that we change the two diagonal elements and change the sign of the two of diagonal elements therefore a inverse is equal to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ upon minus 11 eleven is equal to one by eleven three by eleven three by eleven and minus two by eleven therefore a inverse b is equal to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 11 \\ 3 \\ 3 \end{bmatrix}$ that is $\begin{bmatrix} 11 \\ -3 \\ -3 \end{bmatrix}$ thus we get a unique solution for the system of equations let me now solve another problem suppose it is given that the cost of 4 kg of onion 3 kg of wheat and 2 kg of rice is rupees 60 the cost of 2 kg of onion 4 kg of wheat and 6 kg of rice is rupees 90 and the cost of 6 kg of onion 2 kg of wheat and three kg of rice is rupees 70 find the individual costs

so we represent the system of equations as $\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$ where x is price of per kg of onion y is price of per kg of wheat and z is price of per kg of price we need to find out the values of x y z such that all the three equations are satisfied therefore we need to compute a inverse which is we already know is equal to adjoint of a divided by determinant of a

so we compute adjoint of a a one one is equal to $4 \times 3 - 6 \times 2 = 12 - 12 = 0$ a 1 2 is equal to $-1 \times 2 - 3 \times 6 = -2 - 18 = -20$ a 1 3 is equal to $2 \times 2 - 6 \times 4 = 4 - 24 = -20$ a 2 2 is equal to $-1 \times 3 - 2 \times 3 = -3 - 6 = -9$ a 2 3 is equal to $4 \times 3 - 6 \times 2 = 12 - 12 = 0$ a 3 1 is equal to $4 \times 2 - 6 \times 3 = 8 - 18 = -10$ a 3 2 is equal to $2 \times 6 - 2 \times 2 = 12 - 4 = 8$ a 3 3 is equal to $2 \times 4 - 2 \times 2 = 8 - 4 = 4$

so once we obtain the cofactors we can write adjoint of a is equal to $\begin{bmatrix} 0 & -20 & -20 \\ -9 & 0 & 8 \\ -10 & 8 & 4 \end{bmatrix}$ therefore to compute a inverse we need to find determinant of a now determinant of a is equal to if we expand along the first row we multiplied by the term with their cofactors therefore what we get is $4 \times 0 + 3 \times 30 + 2 \times (-20) = 0 + 90 - 40 = 50$ which is non-zero therefore division by 50 makes sense therefore a inverse is equal to $\begin{bmatrix} 0 & -5 & -4 \\ -9 & 0 & 8 \\ -10 & 8 & 4 \end{bmatrix} \times \frac{1}{50}$ therefore the solution of the equations we can get by multiplying this a inverse with the b vector which is given to us as $\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$ therefore x y z is equal to the product of this with this and that comes out to be $\begin{bmatrix} 0 \\ 9 \\ 14 \end{bmatrix}$ plus 14 is equal to 5 36 minus 28 is equal to 8 minus 24 plus 18 is equal to minus 6 plus 14 is equal to 8.

therefore price of onion is equal to rupees five price of wheat is equal to rupees eight and price of rice is equal to rupees eight are cage let me solve another problem the sum of three numbers is six if we multiply the second the third number and add to the first then we get seven and if we add the second and third number and add that to 3 times the first number then we get twelve find the three numbers

so the above description gives us three equations the first one is $x + y + z = 6$ $x + 2z = 7$ $3x + y + z = 12$ or $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$ therefore as before we first check the what is the determinant of a $\begin{bmatrix} 1 & 0 & -2 \\ 1 & -6 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is equal to $1 \times 0 - 2 \times (-6) + 1 \times (-1) = 0 + 12 - 1 = 11$ not equal to 0 therefore we can compute a inverse is equal to adjoint of a

divided by determinant of a therefore as before we compute the cofactors please check that we get the following a 1 1 is equal to minus two a one two is equal to five a one three is equal to one a 2 1 is equal to 0 a 2 2 is equal to minus 2 and a 2 3 is equal to plus 2 a 3 1 is equal to 2 a 3 2 is equal to minus 1 and a 3 3 is equal to minus 1 again therefore writing it in a matrix form will get adjoint of a is equal to minus 2 5 1 0 minus 2 2 2 minus 1 minus one therefore a inverse is equal to this divided by four is equal to minus two zero two five minus two minus one one two minus one therefore the solution of the equations is equal to a inverse b is equal to one by four into minus two zero two five minus 2 minus 1 1 2 minus 1 multiplied by 6 7 12 is equal to 1 by 4 into minus 12 plus 24 30 minus 14 minus twelve six plus fourteen minus twelve is equal to one by four into twelve 4 8 is equal to 3 1 2 therefore the 3 numbers are 3 1 and 2.

i suggest that you put these values in the equations and you see that all the three equations are satisfied by this three values another way of solving is what is called grammar rule here we wont explicitly compute the inverse rather we shall solve the system of equations by computing different determinants and with the help of that we shall compute the values the idea is if the given system of equations are $Ax = b$ where A is $n \times n$ x is $n \times 1$ b is $n \times 1$ such that A is non singular and b not equal to zero vector that is a $n \times n$ matrix where all values are zeros

so if b is not a zero vector and A is non-singular then the three solutions then the solutions of the three equations can be computed as follows let me illustrate with three by three

so let d is equal to determinant of A let d_1 be the matrix obtained by replacing the first column of A with b vector that is d_1 is equal to determinant of $b_1 \ b_2 \ b_3 \ a_{12} \ a_{13} \ a_{22} \ a_{23} \ a_{32} \ a_{33}$ similarly let d_2 be determinant of the matrix $a_{11} \ b_1 \ a_{13} \ a_{21} \ b_2 \ a_{23} \ a_{31} \ b_3 \ a_{33}$ that is we have replaced the second column of A matrix by the b vector and similarly compute d_3 as determinant of $a_{11} \ a_{12} \ b_1 \ a_{21} \ a_{22} \ b_2 \ a_{31} \ a_{32} \ b_3$ then x is equal to d_1 upon d y is equal to d_2 upon d and z is equal to d_3 upon d i am not proving it but i am verifying it with the same example which we did just now

so verification A is equal to $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$ $x \ y \ z$ is equal to $\begin{pmatrix} 6 \\ 12 \\ 2 \end{pmatrix}$ we knew the answer is three one two

so we solve it using grammar's rule we know d is equal to determinant of A is equal to four d_1 is equal to determinant of $\begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 2 \\ 12 & 1 & 1 \end{pmatrix}$ is equal to $6 \times (0 \times 1 - 2 \times 1) - 1 \times (7 \times 1 - 12 \times 1) + 1 \times (7 \times 1 - 12 \times 1)$ is equal to $6 \times (-2) - 1 \times (-5) + 1 \times (-5)$ is equal to $-12 + 5 - 5$ is equal to -12 therefore the value of the first variable is equal to -12 upon 4 is equal to -3 now d_2 is equal to determinant of $\begin{pmatrix} 1 & 1 & 1 \\ 6 & 1 & 1 \\ 7 & 2 & 3 \end{pmatrix}$ and this we get by replacing the second column with the b vector $6 \ 7$ and 12 .

therefore its determinant $1 \times (7 \times 3 - 12 \times 1) - 1 \times (6 \times 3 - 7 \times 1) + 1 \times (6 \times 1 - 7 \times 1)$ is equal to $1 \times (21 - 12) - 1 \times (18 - 7) + 1 \times (6 - 7)$ is equal to $9 - 11 - 1$ is equal to -3 therefore value of y is equal to -3 by 4 is equal to -0.75 in a similar way d_3 is equal to determinant of $\begin{pmatrix} 1 & 1 & 6 \\ 1 & 1 & 0 \\ 7 & 3 & 1 \end{pmatrix}$ we have replaced the third column with the v vector is equal to $1 \times (1 \times 1 - 7 \times 3) - 1 \times (1 \times 1 - 7 \times 7) + 6 \times (1 \times 7 - 1 \times 1)$ is equal to $1 \times (-20) - 1 \times (-48) + 6 \times (6)$ is equal to $-20 + 48 + 36$ is equal to 64 therefore z is equal to 64 by 4 is equal to 16 therefore we see that using the cramer's rule where we did not explicitly compute the A inverse we could solve the systems of equations all the problems that we solved have the A matrix is non-singular and therefore determinant of A is non-zero question is if A is singular then what happens

so determinant of A is 0 therefore we need to do the following a if determinant of A is equal to 0 and adjoint of A multiplied by b is equal to 0 then we will have multiple solutions for the system of equations and b if determinant of A is

θ and adjoint of a multiplied by b is not equal to θ then there will be no solutions illustration consider two x plus three y is equal to five and four x plus six y is equal to ten.

now a is equal to $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ therefore determinant of a is equal to θ therefore we compute adjoint of a into b is equal to we know that for two cross two matrices to compute that joint we interchange the diagonal elements and we change the sign of the off diagonal elements therefore the adjoint of this matrix is $\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix}$ and if we multiply this with $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ we get 30 minus 30 minus 20 plus 20 is equal to θ θ is equal to θ therefore determinant of a is θ and we find adjoint of a multiplied by b is θ therefore as we have seen there are infinitely many solutions but suppose we have new example two x plus three y is equal to five and four x plus six y is equal to fifteen therefore determinant of a is equal to θ and adjoint of a into the b vector is equal to $\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix}$ multiplied by $\begin{pmatrix} 5 \\ 15 \end{pmatrix}$ is equal to 30 minus 45 and minus 20 plus thirty is equal to minus fifteen ten this is not a zero vector not a zero matrix of size two cross one therefore as we have seen the determinant of a is equal is zero but adjoint of a multiplied by b is non-zero therefore there is no solution to this system of equation or this given system of equation is inconsistent ok students with that i conclude my series of lectures on determinants i hope that you will find the lectures and the examples that i have done useful in solving many other problems thank you
so much you