

welcome students to the fourth lecture on determinants in the last lecture we have solved several problems on computing determinants of three cross three matrices but the techniques developed there can be extended to solve for n cross n square matrices also in today's lecture we shall look into determinants from different angles let us look at first from areas of triangles we know the area of a triangle given by points x_1, y_1, x_2, y_2 and x_3, y_3 is half into x_1 into y_2 minus y_3 plus x_2 into y_3 minus y_1 plus x_3 into y_1 minus y_2 this we have seen in our coordinate geometry question is how we can see it from a determining perspective consider the following matrix one one one x_1 y_1 x_2 y_2 and x_3 y_3

so what we have done we have taken the three points but of course we have taken an extra column 1 1 1 as the first column

so what is the determinant of this matrix if we call it a then determinant of a is equal to $1 \times (x_2 y_3 - x_3 y_2) - 1 \times (x_1 y_3 - x_3 y_1) + 1 \times (x_1 y_2 - x_2 y_1)$ this is equal to $x_2 y_3 - x_3 y_2 - x_1 y_3 + x_3 y_1 + x_1 y_2 - x_2 y_1$ is equal to if i now take x_1 common from these two terms i get x_1 into $y_2 - y_3$ plus i take x_2 to be common from these two terms plus if i take x_3 common from the two remaining terms we get this therefore if we compare with the formula for area of a triangle we get that area is equal to half of the determinant of $1 \times 1 \times 1 \times x_2 \times y_2 \times x_3 \times y_3$ or it is same as half of $x_1 \times y_1 \times 1 \times x_2 \times y_2 \times 1 \times x_3 \times y_3$ one in some books you may get it like that i have done it like this but if we look at carefully we will see that these two values are actually same because here if we swap the first column with the second column we know that 1 1 1 will come here $x_1 \times x_2 \times x_3$ will come here then after that if we swap the second and third column i will get $y_1 \times y_2 \times y_3$ here and one one one here and we know that if we interchange the row two rows or interchange two columns then the determinant gets a negative side since we have to do it twice eventually we will get the determinant with the same sign if determinant comes out to be negative we take the absolute value for area that is obvious because area of a matrix cannot be negative what happens if the determinant is zero area of a triangle will be zero if the points are collinear therefore to test collinearity we may use

determinant in the above way example show that a is equal to a comma b plus c b is equal to b comma c plus a and c is equal to c comma a plus b are collinear therefore we will compute the determinant of one a b plus c one b c plus a one c a plus b now by doing $c_3 \rightarrow c_3 + c_2$ this determinant is equal to $1 \times (a + b) + (b + c) + (c + a)$ plus b plus c into determinant of $1 \ 1 \ 1 \ a \ b \ c \ 1 \ 1 \ 1$ which is 0 as these two columns are same therefore we can see that these three points are collinear as as the triangle formed by this three point as area is equal to zero another example if points a comma 0 0 comma b and 1 comma 1 are collinear then show that a plus b is equal to a b since these are collinear we know determinant of $1 \ 1 \ 1 \ a \ 0 \ 0 \ b \ 1 \ 1$ is equal to 0 by replacing r_1 by $r_1 - r_2$ we get the determinant is $0 \ a - b \ 1 \ 0 \ b \ 1 \ 1 \ 1$ now by replacing r_3 with $r_3 - r_2$ we have the determinant is same as $0 \ a - b \ 1 \ 0 \ b \ 0 \ 1 \ 1$ one minus b now by expanding with the second row first column element we get determinant is equal to minus 1 to the power 1 plus 2 into 1 into a into 1 minus b minus 1 into minus b is equal to minus 1 into a minus a b plus b

so equating with zero we have $a - b + b$ is equal to 0 or $a + b$ is equal to a b this is what we have been ask to show another example use determinant method to find equation of the line joining points a is equal to minus two comma four b is equal to two comma minus six we know from our knowledge of coordinate geometry the equation of the line is $y - 4$ upon $4 + 6$ is equal to $x + 2$ upon $-2 - 2$ or $y - 4$ upon 10 is

equal to $x + 2$ upon minus 4 or $10x + 20$ is equal to minus four $y + 16$ or $10x + 4y + 4$ is equal to zero this scenario case we know but now we do it using determinant and see whether we get the same result so let x, y be any point on the line joining $(-2, 4)$ and $(2, -6)$ therefore determinant of $\begin{vmatrix} 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 1 & 2 & 6 \\ x & y & 1 & 2 & 6 \end{vmatrix}$ is equal to zero or $\begin{vmatrix} 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 1 & 2 & 6 \\ x & y & 1 & 2 & 6 \end{vmatrix}$ this determinant is equal to zero by making r_1 replaced by $r_1 - r_2$ or determinant of $\begin{vmatrix} 0 & -2 & 3 & -1 & -4 \\ 2 & 4 & 1 & 2 & 6 \\ x & y & 1 & 2 & 6 \end{vmatrix}$ is equal to $\begin{vmatrix} 0 & -2 & 3 & -1 & -4 \\ 2 & 4 & 1 & 2 & 6 \\ x & y & 1 & 2 & 6 \end{vmatrix}$ this is by replacing r_2 with $r_2 - r_1$.

now what is the determinant of this matrix this is actually the determinant of these 2×2 sub matrix or $\begin{vmatrix} -4 & -6 \\ 2 & -x \end{vmatrix}$ is equal to 0 or $24 + 4y - 20 + 10x$ is equal to 0 or $10x + 4y + 4$ is equal to zero therefore this is the equation of the line passing through the two given points $(-2, 4)$ and $(2, -6)$ and this is the same result that we have got after applying our knowledge of coordinate geometry

so this way also we can compute the equation of a line passing through two given points let us take another example if the area of the triangle $A(-2, 4)$, $B(2, -6)$ and $C(5, k)$ is 35 units or 35 square units what is the value of k therefore we know that we can get the idea by using determinant that is what is given is that half into $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 4 & 2 \\ 2 & -6 & 5 \end{vmatrix} k$ is equal to 35 or determinant of $\begin{vmatrix} 1 & -2 & 4 & 1 & 2 \\ 2 & 4 & 1 & 2 & 6 \\ 5 & k & 1 & 2 & 6 \end{vmatrix}$ is equal to 70 or determinant of $\begin{vmatrix} 0 & -4 & 10 & 1 & 2 \\ 2 & 4 & 1 & 2 & 6 \\ 5 & k & 1 & 2 & 6 \end{vmatrix}$ is equal to seventy as before I have done r_1 is equal to $r_1 - r_2$ or determinant of $\begin{vmatrix} 0 & -4 & 10 & 1 & 2 \\ 2 & 4 & 1 & 2 & 6 \\ 5 & k & 1 & 2 & 6 \end{vmatrix}$ zero three $k + 6$ is equal to seventy this we get by doing r_3 is equal to $r_3 - r_2$ now we try to find the determinant since we have only a 2×1 is non-zero use determinant is going to be minus 1 times this into this and this into this or -1 into -4 times $k + 6$ minus 3 into 10 is equal to 70 or -1 into -4 $k - 24 - 30$ is equal to 70 or $4k + 24 + 30$ is equal to seventy or $4k$ is equal to sixteen or k is equal to four that is the answer ok students let me now introduce you to the terms minor and cofactors if you remember we have expressed the determinant of a matrix A as $\sum_{j=1}^n a_{1j} \cdot (-1)^{1+j} \cdot M_{1j}$ where M_{1j} is the determinant of the sub matrix of A after deleting r_1 and column j in a similar way we can also write determinant of A is equal to $\sum_{j=1}^n a_{ij} \cdot (-1)^{i+j} \cdot M_{ij}$ by expanding along r_i similarly we can do it by expanding along a column also but what is important is that the term a_{ij} is being multiplied by $(-1)^{i+j}$ into determinant of the matrix which we obtain by deleting the i th row and j th column this M_{ij} is called the minor corresponding to the element a_{ij} therefore definition for a given matrix say if we consider the element a_{ij} that is at the position i th row and j th column then corresponding minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained after deleting i th row and j th column of A also we have seen that in the expansion of the expression for determinant of A we multiply a_{ij} with $(-1)^{i+j}$ into M_{ij} this whole thing is called the cofactor to a_{ij} example consider the matrix $A = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}$ therefore determinant of A is equal to $a \cdot d \cdot \begin{vmatrix} f & g \\ j & k \end{vmatrix} - b \cdot c \cdot \begin{vmatrix} e & h \\ i & l \end{vmatrix} + c \cdot \begin{vmatrix} e & f \\ m & n \end{vmatrix} - d \cdot \begin{vmatrix} e & f \\ i & j \end{vmatrix}$ therefore M_{11} is the minor for this a is the determinant of the sub matrix after deleting the first row and first column is equal to $\begin{vmatrix} f & g \\ j & k \end{vmatrix}$ M_{12} is equal to the determinant of the sub matrix after deleting first row and second column is equal to $\begin{vmatrix} e & h \\ i & l \end{vmatrix}$ M_{13} is equal to the determinant of the sub matrix after deleting column 1 and row 2 is equal to $\begin{vmatrix} e & f \\ m & n \end{vmatrix}$ and M_{14} in a similar way we can get is equal to $\begin{vmatrix} e & f \\ i & j \end{vmatrix}$ but what are the cofactors the cofactors are A_{11} is equal to $(-1)^{1+1} \cdot M_{11}$ A_{12} is equal to $(-1)^{1+2} \cdot M_{12}$ A_{13} is equal to $(-1)^{1+3} \cdot M_{13}$ A_{14} is equal to $(-1)^{1+4} \cdot M_{14}$

equal to d a_{12} is equal to -1 to the power $1+2$ into m_{12} is equal to $-c_{21}$ in a similar way will get is $-b$ and a_{22} is equal to a so that is the subtle difference between the minors and the cofactors let me ask you a simple question if a is equal to a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} which of the following gives determinant of a option a a_{21} into m_{21} plus a_{22} into m_{22} plus a_{23} into m_{23} that is i am trying to expand it along the second row but we find that the signs have not been taken care of here we should have been e^{-1} to the power $2+1$ therefore there should have been a minus here there should have been a minus here therefore this is not correct option b a_{11} into m_{11} minus a_{12} into m_{21} plus a_{13} into m_{31} as we can see it has taken care of the sign here it is going to be positive because $1+1$ here it is going to be positive because $1+3$ these are even

so -1 to the power an even number makes it one but here it is one and two therefore the minus sign is there but here the problem is a_{12} is multiplied by the cofactor m_{21} which should have been m_{12} therefore this is also wrong option c a_{31} into a_{31} minus a_{32} into a_{32} plus a_{33} into a_{33} here instead of minors we are using cofactors therefore the minus sign is already taken care of inside the cofactor therefore this minus is wrong therefore this also does not give us the correct value of the determinant option d is a_{13} into a_{13} plus a_{23} into a_{23} plus a_{33} into a_{33} if we look at that we can understand that we are expanding along the column number 3 and each term has been suitably multiplied by corresponding cofactors therefore this is the right answer therefore answer is option d let me now introduce a new concept which is called adjoint of a matrix definition the adjoint of a square matrix a is defined as the transpose of the matrix a_{ij} where a_{ij} is the cofactor of a_{ij} that is if a is equal to a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} then adjoint of a is the three cross theory matrix which is a_{11} a_{21} a_{31} a_{12} a_{22} a_{32} a_{13} a_{23} a_{33} because we are writing as the transpose a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} example if a is equal to a b c d then a_{11} is equal to d a_{12} is equal to $-c$ a_{21} is equal to $-b$ and a_{22} is equal to a therefore adjoint of a is the matrix d $-c$ $-b$ a that is the diagonal elements are interchanged and sign of off diagonal elements are changed thus for a 2 cross 2 matrix we get the adjoint very easily now what is the product of a times adjoint of a

so let us do it we have a is equal to a b c d and adjoint of a is equal to d $-c$ $-b$ a and if i multiply this we get a two cross two matrix which is a d $-b$ $-c$ $-a$ b c d $-c$ d and $-c$ b a d is equal to determinant of a θ θ determinant of a that is it is a diagonal matrix with diagonal entries equal to determinant of a therefore for our 2 cross 2 matrix we see that a into adjoint of a is equal to determinant of a into i which is the identity matrix of size two cross two in general if a is n cross n then also we find a into adjoint of a is equal to determinant of a into a into an identity matrix of size n cross n let me verify the result for 3 cross 3 matrix therefore let p be the product of a into adjoint of a that is p is equal to a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} into a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} what is the value of p as i said it is going to be the determinant of a into i three we want to verify that p is equal to determinant of a into one one one zero zero zero zero zero zero let us look at the p one one if we look at this it is a_{11} into a_{11} a_{12} into a_{12} and a_{13} into a_{13} p_{11} is equal to a_{11} into a_{11} plus a_{12} into a_{12} plus a_{13} into a_{13} which we know that is the expression of the determinant when we expand along the first row of a therefore this is going

to be determinant of a similarly $p \times p$ and $p \times p$ can be shown to be determinant of a we need to show that off diagonal elements are zero i verify with $p = 2$

so what is $p = 2$ $p = 2$ is equal to $a_{11}a_{22} + a_{12}a_{21} - a_{13}a_{23} - a_{31}a_{32} - a_{33}a_{11} - a_{33}a_{22}$ because this we get from the adjoint of $a_{21} + a_{12} - a_{13}a_{31} - a_{32}a_{33}$ corresponding to a_{22} and therefore determinant of $a_{11}a_{13}a_{31} + a_{33}a_{11} + a_{12}a_{13}a_{33} - a_{12}a_{11}a_{33} - a_{13}a_{11}a_{32} + a_{13}a_{31}a_{12}$ now let us expand it this is coming out to be $-a_{11}a_{12}a_{33} + a_{11}a_{13}a_{32} + a_{12}a_{11}a_{33} - a_{12}a_{13}a_{31} - a_{13}a_{11}a_{32} + a_{13}a_{31}a_{12}$ now let us look at the terms we find that $-a_{11}a_{12}a_{33} + a_{12}a_{11}a_{33}$ and this is $+a_{12}a_{11}a_{33}$ therefore they cancel each other this is $a_{11}a_{13}a_{32} - a_{13}a_{11}a_{32}$ and this is $-a_{13}a_{11}a_{32} + a_{13}a_{31}a_{12}$ therefore they also cancel each other this is $-a_{12}a_{13}a_{31} + a_{13}a_{31}a_{12}$ therefore this also cancels with this therefore this whole thing is equal to zero

so this i have verified for $p = 2$ i suggest that you convince yourself by working out on other of diagonal elements and computing their values which we will see that will come out to be zero therefore a into a joint of a is equal to determinant of a into i three okay students i stop here today in the next class i will talk about inverse of a matrix and then i will show certain properties of it and how we can use it in solving system linear equations ok students thank you