

welcome students to the first lecture on determinants i assume you are all familiar with matrices in earlier lectures of iit paul we have discussed matrix in great details

so i assume that you know that a matrix is a rectangular array of real or complex numbers if A is a matrix then we denote it by say a_{11} a_{12} up to a_{1n} a_{21} a_{22} a_{2n} like that a_{m1} a_{m2} a_{mn} where

so the order of the matrix is m cross n that is there are m rows and n columns and a_{ij} denotes the element at i th row j th column position we denote such a matrix as a_{ij} i is equal to one to m j is equal to one to n where each a_{ij} is a real or complex number i further assume that you know matrix addition subtraction multiplication and transpose if you do not remember this please go back and revise this how to carry out these operations on a matrix for this series of lecture on determinants i assume that you are familiar with this and with that background i will start what is called a determinant corresponding to each square matrix we associate a number which is called its determinant

so what is the square matrix a square matrix is one in which the number of rows is same as the number of columns or a square matrix can be denoted as a_{ij} i is equal to one to n and j is equal to one to n that means both the number of rows and number of columns of the matrix are equal in that case we call it a square matrix how is determinant defined if A is of order 1 cross 1 that means if A has only one row and one column that is A is a constant say a then its determinant is a if A is two cross two that is there are 2 rows and 2 columns then we can write A as a_{11} a_{12} a_{21} a_{22} and its determinant we should denote as like this is equal to $a_{11}a_{22} - a_{21}a_{12}$ that means it is the difference of the product of the diagonal elements and the product of the off diagonal elements for example A is equal to $1\ 2\ 3\ 4$ then determinant of A is equal to one into four minus three into two is equal to four minus six is equal to minus two another example A is equal to $a\ b\ c\ b\ c$ minus $b\ a$ minus b if this is the matrix these are four different scalars then determinant of A is equal to $a^2 + b^2 - c^2 - b^2$ which is equal to $a^2 - c^2$ a third example A is equal to $\cos\theta\ \sin\theta\ \sin\theta\ \cos\theta$ for some arbitrary θ then determinant of A is equal to $\cos^2\theta - \sin^2\theta$ is equal to $\cos^2\theta + \sin^2\theta$ is equal to one one last example with two cross two if A is equal to $1\ 2\ 2\ 4$ then determinant of A is equal to $1 \times 4 - 2 \times 2$ is equal to $4 - 4$ is equal to zero once we know the determinant of two cross two matrices we can develop the determinant for higher order matrices for example consider now A which is 3 cross 3 matrix that means there are 3 columns and three rows

so let A be a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} or a_{31} a_{32} a_{33} in this case determinant of A is equal to will expand it around the first row so i consider a_{11} and this is to be multiplied by the determinant of this sub matrix that is determinant of a_{22} a_{23} a_{32} a_{33} minus a_{12} i am now taking the term first row second column as i said i am expanding the determinant along the first row of the matrix and this will be multiplied by the sub matrix which we can identify as follows since i am taking a_{12} i wont consider this and do not consider this column

so what is remaining is this sub matrix a_{21} a_{23} a_{31} and a_{33} and therefore minus a_{12} will be multiplied by a_{21} a_{23} a_{31} a_{33} this sub matrix determinant plus now only one more element is left that is a_{13} therefore a_{13} times determinant of the sub matrix that we get after deleting the first row and the third column therefore this is going to be you can easily see is determinant of a_{21} a_{22} a_{31} a_{32} which now i can expand

as a_{11} times a_{22} into a_{33} minus a_{32} a_{23} minus a_{21} a_{12} into a two one into a three three minus a_{23} into a three one plus a_{13} into a two one into a three two minus a_{22} into a three one if i expand it further i get it is $a_{11} a_{22} a_{33}$ minus $a_{11} a_{32} a_{23}$ minus $a_{12} a_{21} a_{33}$ plus $a_{12} a_{23} a_{31}$ plus $a_{13} a_{21} a_{32}$ minus $a_{13} a_{22} a_{31}$.

so this is going to be the determinant of the 3×3 matrix which we denote by a_{ij} i is equal to one to three and j is equal to one two three example a_{11} is equal to one two three 4 5 6 3 1 2.

so what is going to be is determinant a_{11} which is 1 multiplied by the determinant of this sub matrix which is 5 into 2 minus 6 into 1 minus 2 into because a_{12} is 2 2 into the determinant of this sub matrix 4 into 2 minus 3 into 6 plus this is 3 therefore 3 times 4 into 1 minus three into five is equal to one times ten minus six minus two times eight minus eighteen plus 3 times 4 minus 15 is equal to 4 plus 20 minus 33 is equal to 24 minus 33 is equal to minus 9 that is the determinant of this 3×3 matrix we will work on computing for others many other matrices in this class but let me first mention a few points one is it necessary to expand only along row one as you have seen i have computed the determinant by taking terms from the row and by multiplying it with the determinant of a sub matrix and putting the sign plus or minus as per the formula therefore natural question is do you always need to go along first row the answer is no in fact we can expand it along any row or any column but we have to be careful in choosing the right sub matrix and the sign as you have seen in some of the terms we put plus sign in some of the terms we put minus sign how that can be determined the idea is as follows if we are expanding along the i th row then determinant of a is equal to let me write a_{i1} multiplied by the determinant of a sub matrix let me call it m_{i1} and there will be a sign minus 1 to the power $i + 1$ plus minus 1 to the power $i + 2$ a_{i2} that is the second element of the row multiplied by the sub matrix determinant of the sub matrix m_{i2} where m_{i2} is obtained by deleting the i th row and the second column from the original matrix like that we go and finally we will write minus 1 to the power $i + n$ a_{in} determinant of the sub matrix m_{in} which can be obtained by deleting the i th row and the n th column of the matrix a

so the formula is clear at each point depending upon along which row we are expanding we have to take the corresponding sub matrixes and they have to be multiplied by the terms but the sign of that will be determined by the row number and the column number that element of the element on which we are expanding now and it is going to be minus 1 to the power $i + 1$ in general minus 1 to the power $i + j$ which is going to be the sign of the product of a_{ij} multiplied by the determinant m_{ij}

so let me illustrate it illustration consider the same matrix one two three four five six three one two and we computed determinant of a is equal to minus nine by expanding along the first row let us expand along the first column therefore determinant of a is equal to one multiplied by five into two minus one into six plus minus 1 to the power 2 and 1 multiplied by the element 4 multiplied by the determinant of the sub matrix obtained by deleting second row and first column therefore this is going to be two into two minus three into one plus minus 1 to the power this element it is the third row first column element

so it is 3 plus 1 multiplied by 3 into 2 into 6 minus 3 into 5 is equal to 1 times 10 minus 6 minus 4 times 4 minus 3 plus 3 times 12 minus 15 is equal to 1 into 4 minus 4 into 1 plus 3 into minus 3 is equal to 4 minus 4 minus nine is equal to minus nine let me now expand along the third column we have a is equal to one two three four 5 6 3 1 2 and we are expanding along the third column

therefore determinant of a is equal to minus 1 to the power this is first row third column one plus three into determinant of this sub matrix four into one minus three into 5 plus minus 1 to the power 2nd row and 3rd column and the element is 6 and i am multiplying it by the determinant of this sub matrix one two three one therefore it is going to be one dot one minus three dot two plus minus one to the power third row third column that element is two multiplied by one dot five minus 2 dot 4 is equal to this is 1 therefore 4 minus 15 this is minus 6 into 1 minus 6 plus 2 into 5 minus 8 is equal to minus eleven sorry i miss the three here the element therefore this has to be multiplied by three therefore this is going to be minus thirty three it is minus 5 minus 6

so it is plus 30 it is minus 3 into 2 therefore minus six is equal to minus nine therefore we can actually see that by expanding it with different row and different column we are getting the same answer in order to visualize it for a general matrix consider a three by three matrix a is equal to a 1 1 a 1 2 a 1 3 a 2 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3 its determinant when expanded along row one is as we know a 1 1 into a 2 2 into a 3 3 minus a 2 3 into a 3 2 minus a 1 2 into a 2 1 into a 3 3 minus a 2 3 into a 3 1 plus a 1 3 into a 2 1 into a 3 2 minus a 2 2 into a 3 1.

which is is equal to a 1 1 a 2 2 a 3 3 minus a 1 1 a 2 3 a 3 2 minus a 1 2 a 2 1 a 3 3 plus a 1 2 a 2 3 a 3 1 minus a 1 3 a 2 2 a 3 1 plus a 1 3 a 2 1 a 3 2 minus a 1 3 a 2 2 a 3 1 and we have minus a one three a two to a three one therefore we find that instead of expanding it along the row one if we have expanded it along the row 3 we get the same answer for your conviction i shall now expand along the second column we had a one one a one two a one three a two one a 2 2 a 2 3 a 3 1 a 3 2 a 3 3 therefore determinant of a expanding along second column is minus a one two since second column is this and i am starting with a one two one plus two is three which is odd number therefore minus one to the power three is equal to minus the minus a one two into a two one a three three minus a three one into a two three plus a 2 2 into a 1 1 a 3 3 minus a one three a three one minus a 3 2 into a 1 1 into a 2 3 since i am using a three two now the determinant of a one one a two three minus a one three a 2 1 which when we expand we get minus a 1 two a two one a three three plus a one two a two three a 3 1 plus a 1 1 a 2 2 a 3 3 minus a 1 3 a 2 2 a 3 1 minus a 1 1 a 2 3 a 3 2 plus a 1 3 a 2 1 a 3 2 let me now compare the results with the one that we got by expanding along the first row we have a one one a two two a three three here also we have a one one a two two a three three minus a one one a two three a three two minus a 1 1 a 2 3 a 3 2 minus a 1 2 a 2 1 a 3 3 minus a 1 2 a 2 1 a 3 3 plus a 1 2 a 2 3 a 3 1 plus a 1 2 a two three a three one plus a one three a two one a three two plus a one three a two one a three two and finally minus a 1 3 a 2 2 a 3 1 minus a 1 3 a 2 to a 3 1 thus we see that when we expanded along the first row or the third row or the second column we get the same answer for any general matrix a this is not a proof but you can verify in a similar way that whether you expand it along row 2 or column 1 or column 3 you will get the same expression therefore the

not writing because it is multiplied by 0 plus this 0 times this determinant but that is the material because it is being multiplied by 0 therefore what we are left with is a 1 1 a two two a three three therefore when we are given any diagonal or a triangular matrix we wont have to bother calculating the determinant by expanding with some first row or first column we will just look into it and we understand that because it is a triangular matrix its determinant is going to be the product of the diagonal elements property four if we multiply each element of a row or column by a constant say k then the determinant will also be multiplied by k or in other words if a is equal to a 1 1 a 1 to a 1 3 a two one a two two a two three a three one a three two a three three and we get a new matrix b say by multiplying the first row with k then we are elements of b are k 1 1 k a one two k a one three a two one a two two a two three a three one a three two a three three then determinant of b is k one one k a one one into a two two a three three minus a 2 3 a 3 2 minus k a 1 2 into a 2 1 a 3 3 minus a 3 1 a 2 3 plus k times a one three multiplied by a two one into a three two minus a two two into a three one thus we see that all the three terms are multiplied with one k

so if i take k to be common then what i get is basically a one one into minus a two three into a three two minus a one two into a two one a three three minus a three one a two three plus a one three into a two one a three two minus a two two a three one and this is nothing but the determinant of a therefore this is k times determinant of a ok friends i stop here today in the next class i shall explore a few more properties and will try to solve some more problems with determinants thank you you