welcome to this lecture on the properties of determinants

so this is the second lecture in this ah series on determinants previously we had talked about the definition of determinants and some of the areas some of the instances where it comes up ah next we are going to talk about the properties of determinants properties of determinants

so the idea behind trying to calculate the properties of the determinants is just like with any other property that we may have seen for example the distributive property of multiplication ah the main idea is to add to what we already know from the definition by trying to say ok can we simplify the computation of certain quantities

so what we want to do is to say ok we know the definition of a determinant but can we come up with some properties which will simplify or make more efficient the process by which we compute these determinants ah for example if you think about the distributive property that you have three scalars a b and c and then you say that ok a times b plus c the sum b plus c is nothing but taken the sum individually a times b plus a times c now even though this is something which follows from the definitions of multiplication and addition what we can see and we will just shortly write it down is that the idea is that in one way one is able to compute the same quantity in a more efficient way

so what i mean is the following

so the idea is as follows

so consider three numbers a b c which are let us say scalars we know that a times b plus c is equal to a times b plus a times c

so these two are equal

so in principle given any three numbers either we can compute using this or we could compute using this but there is a little bit difference in the efficiency in the two processes

so for example if we compute using this way we have one multiplication that we have to do and we have one addition that we have to do whereas on this side we have to do two multiplications and one addition

so even though these are the same quantity we have to apply different amount of effort

so same quantity but different effort

so that is the idea behind ah looking at these properties of the determinants using the definition of the determinant we can anyway calculate the determinant but by exploiting some features in the determinants the definition ah we can come up with some simplifying principles which are of course ah not only useful when we try to compute the determinants but also when we want to make a computer evaluate certain determinants especially when the size of the the square matrices for whom we are interested in calculating the determinant it becomes larger

so that is the basic idea of looking at the properties of the determinants

so we will start by looking at the properties um and try to verify the properties using some examples ah try to get an idea of why certain properties make sense ah and

so on ok

so properties of determinants determinants

so we will start off by just noting certain simple determinants the values of determinants of some simple matrices the idea would be that ok if you can identify certain matrices or certain kinds of matrices which have relatively simpler determinant calculations then what we could do to calculate the determinant of a more relatively complicated matrix is to try to ah bring it closer in the form using these properties

so the first property or the first relatively easy determinant that we can

notice is just coming directly from the definition and that is that if you have a square matrix with an entire row of zeros ok then what would be the determinant of that matrix it would be nothing but 0 why is that because just from the definition if you expand along the row which has 0 entries the sum of the products of the entry of the row and its corresponding cofactor each of them will be 0 and
 so the overall determinant would be zero
 so the first property that we can write down is that if the entire row of a determinant is zero ok
 so excuse me this should be if the entire row of a square matrix is zero then the value of its determinant is also zero
 so what this is what we are trying to say is that if the entire row of a square matrix is zero then the value of the determinant is also zero now ah as we had seen in the definition of a determinant it is not just that we can have expand the square matrix along row but we can also do it along a column and
 so ah in completeness this property should read if the entire row or column of a square matrix is 0 then the value of the determinant is 0.

 so let me just make that addition here
 so i say that if the entire row or column of a square matrix is zero and this kind of an interchangeability between the row and the column is something that we will see in other properties as well
 so it is good to make note of it here and now let us just look at a two by two matrix to see how this works out
 so if the example that we consider is a two by two matrix which has two zero entries c d then either we take the determinant using um the definition or the formula that we know that this is this entry time this entry minus this in either way we can get that if we expand along this row it is 0 times something plus 0 times something else and because each of them is 0 the sum is zero and one can imagine doing this for a general n by n square matrix just expand along a row or expand along a column if if it is entirely zero then the value of the determinant is going to be zero ok
 so this is a simple property but something that is good to make a note of okay
 so the next property that we want to talk about property two the property two also has a similar flavor and the idea here is that if you have a diagonal matrix
 so recall a diagonal matrix is a matrix which has elements only along its diagonal and the remaining entries are zero
 so if we look at a diagonal matrix then its determinant is nothing but the product of the diagonal entries
 so for a diagonal matrix diagonal matrix the determinant is the product of the diagonal entries and why is this the case well again this follows directly from the definition in that if you try to now expand along any row or column ah what you will be left with because there is only one entry which will be non-zero
 so that will be the only term that you get multiplied by its cofactor which will always
 so be have a structure similar to the structure of a diagonal matrix
 so let us just take a look at a three by three matrix example to get an idea of this
 so the example that we look at of a 3 by 3 matrix is that if you have a matrix a 1 it's a diagonal matrix right
 so the off diagonal terms are zero a two two a three three ok now if this is a matrix and we want to take the determinant of this then we can expand along any row or column

so let us for example expand along this row

so this is going to be equal to this entry a 1 1 times its cofactor

so the cofactor will be obtained remember from the definition by deleting this entire row this column and then what we are left with ah a 2 2 0 0 a 3 3 this would be the cofactor because the index of this 1 1 is such that minus 1 power 1 plus 1 which is nothing but 1 and then the other terms we do not have to worry about because they are 0s because these terms are 0s now what about this term again either directly from the definition or by knowing the determinant of a two by two matrix we can just write this down as a one one times a two two times a three three

so there we have it that the determinant of a diagonal matrix is nothing but the product of the diagonal elements and what we have illustrated is for a three by three matrix in general for 2 by 2 or more generally any other kind of order n by n we can have the same sort of property that if you look at a diagonal matrix its determinant if you want to calculate it is very easy you just have to look at the diagonal terms make a product of them and that is a determinant and to show it in general also we could do this sort of ah expanding along a row or column and again just like for the previous property it is good to know this because if you have a determinant which has a something similar form like a diagonal matrix then we know that it is going to take a an easier way to evaluate the determinant and not just for a diagonal matrix and in fact this is the the topic of property three which is that if you have any triangular matrix a triangular matrix is something which has elements only on one side of the diagonal either upper triangular or lower triangular but in those cases also and by using roughly the same procedure we can show that the determinant of a triangular matrix will also be the the product of the diagonal entries

so let me write this down let me write down this property

so property three is that for a triangular matrix the determinant is the product of diagonal entries as an example let us consider a simpler two by two example just a matrix which has a b c and there is a 0 here

so this is in the form of a upper triangular matrix right and what is the determinant of this

so the determinant of this is nothing but a times c a type c which is just the product of the diagonal entries we could also look at a three by three or a general n by an example but the idea is that if you have a general matrix let us say in this particular case three by three

so if you have a b c and in this case let us look at a three by three lower triangular matrix

so we have zeros in these cases and then it does not matter what is over here it could be anything right

so this is the lower triangular matrix if you would like to evaluate the determinant of this matrix as before we can expand along this row because these two entries are zero

so that is going to be a times its cofactor

so cofactor will have b zero ah dot c

so this dot represents it could be anything we do not need to worry about these two terms because they are zero and in expanding this determinant we know it is again abc

so again this is the product of diagonal entries of diagonal entries right

so this is again ah pointing out how to calculate the determinant of ah a matrix for which calculation of the determinant is easy ah relatively speaking

so if you have a triangular matrix or a diagonal matrix or a matrix which has a one whole row or column of zeros then it is relatively easy to calculate its determinant

so in this set of three properties what we have just seen or just noted are ah broadly speaking easily calculable determinants all right um
 so next we will move on to the next set of properties ah again just to point out over here that we could also try to discuss how these properties are related to some of the geometric ideas or the algebraic ideas that we discussed when we mentioned about the determinant for example if you think about a ah matrix or square matrix which has an entire row of zeros right
 so if you have let us say two by two matrix because that is the dimension in which we discussed about the geometry of a determinant and one row is zero
 so what it means is that the two columns of the matrix are aligned along the y axis
 so basically there is no area and that makes sense that the determinant is 0 because there is no area enclosed by the parallelogram
 so let me just briefly illustrate this idea
 so this is a small note and that is that if you have a 2 by 2 matrix with one row as zero and
 so this could be anything let us say a and b and then if you try to sketch this geometrically like we did in the last lecture this is the x axis and this is the y axis if you think about these as the columns the vector column vectors
 so one is 0 a
 so maybe this is 0 a and maybe the other one is zero b
 so what is the parallelogram over here there is no parallelogram because these two vectors are parallel in fact they are collinear therefore the area enclosed by the columns column vectors is 0 and this is consistent with the determinant itself being a 0.
 now the fact that the determinant is zero you could again calculate it directly or you could just note from the property that ok you have a matrix full of zeros and
 so the determinant is zero
 so ah the main idea for just making this small note is that for the property that we just discussed or for those that we may discuss for some cases it might be helpful to just have a geometrical picture in mind because it just adds a layer to our understanding all right ok
 so now next we want to move to the next property
 so property four
 so this property relates to the determinant of a matrix and its transpose
 so recall when we talk about the transpose of a matrix the transpose of a matrix is obtained by interchanging the rows and their columns and something which comes directly from the definition of the determinant is that ah they have the same determinants
 so matrix and its transpose have the same determinant
 so the property is that if we interchange the row and column of a matrix square matrix the interchanging the row and the column of a square matrix ah does not change the value of determinant in other words the determinant of a square matrix a
 so we are using determinant here as a notation for calculating the determinant of a square matrix is equal to the determinant of its transform
 so a transpose is a symbol used for denoting the transverse of a matrix
 so determinant and its transpose have the same ah sorry apology
 so the matrix and its transpose have the same determinant and this is something that you can see for one dimensional or a one by one square matrix
 so that is just the transpose is equal to the matrix
 so nothing new about it ah for a two by two matrix also we can calculate and see this happens

so let us look at that

so if you look at a 2 by 2 matrix you have a matrix a b c d ok what is its transpose the transpose of this or what we obtained by interchanging the rows in the columns

so here a b is a row

so we can make that a column a b again c d is a row we can make that a column c d ok

so this one is a d minus b c is a determinant

so the determinant of this is a d minus b c what about here again the determinant of this is a d minus b c

so here also the determinant is a d minus b c

so these two are the same

so you can take a transpose of a matrix and you can get the same determinant this is for a two by two matrix for three by three also we can ah sort of verify or check how it follows from the determinants definition because there we say okay we can expand along a row or a column

so if you want to see that here rather than calculating explicitly the value of the determinant we could say ok instead of expanding along the row a b instead of expanding along the row a b here we are expanding along this column a b and then we see that the cofactors come out to be the same and

so the values of the determinants come out to be the same for a more general matrix three by three also we could apply the same logic you have the first row becomes the first column if you expand along that then ah iteratively or using induction one could say ok because we know a two by two matrix and its transpose has the same determinant therefore the cofactor matrices the matrices which form the cofactors also have the same determinant and

so on

so we could do this for a general environ square matrix ah but the point is just and the point to note over here is that a matrix and its transpose have the same determinant all right

so this is the property 4 which you wanted to talk about now the next property has to do with what happens when you interchange two rows of a determinant

so if you have a general matrix square matrix and you change one row and exchange it in another row then what happens to the value of the determinant similarly for a column if you have one column you interchange it with another column what happens to the value of the determinant and what we will see is that if we do this the sign of the determinant changes

so let me write it down and then we will see why that happens

so the next property is that if any two rows or columns

so again there is this kind of a duality between rows and columns in the sense that what we say about one applies equally well to the other and that is sort of following because the matrix and a transpose where you interchange the rows and columns have the same determinant

so if they are interchanged then the sign of the determinant changes um changes

so let me read this again if any two rows or columns are interchanged then the sign of the determinant changes right and just to see this in operation we can consider some examples ah let us start with a two by two example the idea would be that we will replace a row and a column and then see what happens to the value of the determinant all right

so in this example and this example will also help us to just establish a notation for these changes

so we have our two by two matrix a b c d let us call this rho one let us call this rho two and the transformation that we are doing is that we are interchanging r one and r two all right

so what happens when we interchange r1 and r2 instead of row 1 a b we have rho 2 which is c d and instead of c d because we have interchange it with a b we have a b now let us calculate the determinants what is the determinant of this again as before for two by two matrix a d minus b c or we could do it by expanding along a row and finding out the cofactors
 so this is a d minus b c what about over here again using the formula we have here the first entry in the row one column one is c
 so it is actually c b minus a d right and if you look at these two terms this is nothing but minus of a d minus b c
 so the sign between these two if we compare the sign changes all right
 so if you interchange a row or we could equally do it for column we could do it for a two by two matrix that we have seen here we could do it for three by three four by four in general for an n by n matrix ah we can establish over here we have just verified but more formally also this can be said that if you just interchange the rows in the columns then the sign of the determinant is going to change the absolute value will be the same but the sign will change all right
 so this is the property phi ah the next property is interesting ah it uses this particular property to prove it but what it says is that if you now have two rows which are identical then the value of the determinant is also zero ah just just ah very remarkable sort of a property and especially when we look at it the proof of that is also very fascinating and this statement is not just confined to that of a row as we have seen whatever we say about a row in most cases we can also say about the column and the statement there would be that if two columns are identical then the determinant of that matrix is also 0.

 so let us just look into that
 so the property six is if two rows or columns of a matrix are identical
 so identical means that they are same then the value of its determinant is zero ok
 so if two rows or columns of a matrix are identical identical then the value of its determinant is zero right
 so but how do we show this of course we could look at some examples and say but more interestingly we could use a property five how well let us say we have a general square matrix a if you believe property phi then we have that if we interchange any two rows or columns then the sign of the determinant should change ok but if two rows are identical if you interchange them the matrix itself is not going to change but the sign of the determinant will change
 so this is only possible if the value of the determinant itself is 0.

 so let me just write this down
 so the proof is
 so consider a square matrix a right which has identical rows with two identical rows let us call those rows r i and r j ri energy we know that interchanging the rows r i r j will give same matrix a will give same matrix a but from previous property the sign of the determinant changes that is that if you had determinant a was the determinant of the matrix a if you changed the value of the interchange the rows r i and r j then the sign of the determinant would be minus of this
 so determinant of this would be the determinant of the matrix a where the sign rows r i and r j are interchanged but because r i and r are identical this matrix is nothing but a therefore we come to a situation where minus determinant of a is equal to determinant of a which is only possible if the determinant of a is zero all right
 so using the previous property we want to say here that if you have a square

matrix a which identical rows two identical rows or we can say have the same argument for two identical columns then the value of that determinant is going to be zero ok

 so this is also a interesting property because it has many ramifications or many corollaries that we could look at one of them is interesting

 so let me just make a note of that and that goes back to the definition of a determinant

 so if you think about the definition of a determinant it says that a determinant is nothing but the sum of the product of rows of entries of one row or column and their corresponding cofactors what happens if you take the sum of the product of entries of a row with the cofactors of another row what i am saying is the following let us look at a three by three matrix and let us say that we have a three by three matrix consider a three by three matrix as ah as before we use a one one a one two ah a one three a two one a two two a two three a three one a three two and a three three all right

 so these a small a's determine the entries of a matrix and let us say that capital a is cofactor of the entry a i j then determinant is nothing but let us say if you expand along a one one then the determinant is going to be a one one times a one one plus a one two times a one two plus a one three times a one three right now the question is what happens if you replace these cofactors with the cofactors of these other elements in other words what happens if i do the following a one one is a two one plus a one two times a two two plus a one three times a two three

 so what is this value and in general we could ask this for other expansions of rows or columns what happens over here well the idea is or the answer is that this is going to be 0 and an easy way to look at this would be to try to go back from this sum of products to the matrix whose determinant is expressed by this expression it will turn out that that matrix has two identical rows and just from the previous property that we saw if you have something which has two identical rows then that matrix determinant is going to be 0 let me write that down

 so that expression a 1 1 times a 2 1 plus a one two times a two two plus a one three times a two three what is this now let us just expand ah a two one this is a one one a two one is nothing but the cofactor of the element a two one

 so there we have minus one power two plus one times the determinant of the matrix obtained by deleting the entire row containing a21 and the column

 so this is a12 a13 and then a 3 2 a 3 3 okay plus a 1 2 minus 1 power 2 plus 2 times the determinant of the matrix obtained by changing the entire row or deleting the entire row and column of a two two

 so that is a one one a one three a three one a three three and similarly the last term is going to be a one three minus one power two plus three times a one one a one two a three one a three two ok

 so this term is minus one this term is plus one and this term is minus one

 so if i take minus one outside then if i write this down this is going to be a one one times a one two a one three a three two a three three minus a one two into a one one a one three a three one a three three right plus this term last term which is a one three times a one one a one two a three one a three two ok and then i close the bracket

 so if you now just inspect this this will nothing but the determinant of a matrix

 so we are sort of going reverse the definition of a determinant

 so if i just say that this is a determinant of a matrix a 1 1 a 1 2 a 1 3 a 1 2 a 1 1 a one two a one three and a three one a three two a three three which was

 so the step from here to here was by just expanding along this row and what we

see here is that there are these two identical terms and just from the previous property we know that this means that this determinant is 0.

 so therefore what this is telling us if i can just draw this big arrow here is that for evaluating a product of the entry of a row and multiplying it with the cofactors of another row the value of that sum is going to be zero
 so this is an interesting property
 so i mean if you take the sum of the product of entries of a row and their cofactors ah that is the determinant but if you now take in replace their own cofactors with the cofactors of a different row then that sums to zero and that can be shown using this property all right
 so that is this now what we want to do is
 so this is the next set of three properties where we have talked about what happens if you interchange the rows what happens to the transpose if you interchange the rows in the columns what happens if something is identical rows or columns then we have come up with simple ways to say what the how the determinant of that matrix changes and with practice one these properties become second nature
 so you can look at some operation and see what happens to the value of the determinant now the next set of three properties are also interesting they have to do with what happens if you take some sums or products of the either this the entries of the matrix
 so what happens if for example you multiply the entire ah row of matrix with some number how does the determinant change or if we ah sum two rows of matrix how does the matrix change right
 so these are things that we will look at next these are next set of properties
 so the immediate next set is property 7 and this says that if we multiply if each element of a row or a column is multiplied by a constant a constant let us call it k then the value of the determinant also gets multiplied by k
 so the idea is that if you take the entire row change its value by multiplying every row element by k or for a column then the value of the determinant of multiplying
 so with k and um how do we show this
 so one way by looking is just by looking at the definition of a determinant
 so you consider expanding along the row which is getting multiplied ah that before and after the multiplication each entry of the row has increased by a factor of k and
 so we can take that factor out and then we get that the determinant itself is increased by k
 so we can look at a let us say two by two matrix example ah to see the case
 so if we say a matrix a b c d and this element r one the row r one goes to k times r one ok
 so the matrix becomes k a and k b
 so the question is what happens to their determinants
 so the determinant over here and if instead of looking at the formula a d minus b c let us just write it down as a times its cofactor whatever that is we know it is d but let us just write it down plus b times its cofactor we know it is minus c but let's just write it down
 so this is the sum ok
 so here actually should be c but we really do not ah sorry
 so this should be d and this should be minus c but we really do not worry about this
 so much the main idea is you expanded along the row a b now here if we also expand along the first row what is the determinant it is going to be k a times

the cofactor which does not change its the same d plus k b times the cofactor which again doesnt change minus c
 so here this k can be just taken out
 so this is going to be k times a times this plus b times whatever is the cofactors minus c here t here and you see that the expressions here are the same as this
 so these two are the same only thing is k
 so the determinant goes up the determinant scales by the factor k
 so this is just if you do some scalar multiplication of an entire row and we can do this for a column also what we will see is that the value of the determinant is going to ah change or scale by a factor k all right
 so this is a property regarding multiplication and now we want to see what happens if you have a sum involved in the matrix what happens to the determinant
 so to be specific you are looking at a property ah of a matrix that if every entry can be expressed as a sum of two elements
 so if every entry of a row can be expressed as a sum of two elements then the determinant can be written down as a determinant of matrices which have been found by separating the sums
 so let me just write it down it may be ah more clear
 so the property statement is that if some or all elements of a row or column can be written as a sum of two terms then the determinant the overall determinant can be expressed as the sum of determinants and this sum of determinants are of matrices obtained by separating that original matrix separating the original matrix what does this mean
 so let us just look again at a simple example again a two by two example suppose we have a matrix a plus x let us say and b plus y all right and then c and d
 so what is the determinant of this well we could try to say ok i should point out that this x because it says sum or all even if you cannot write it as a sum in some cases this x could be just ah zero
 so it is not that we need all of them to be written as sum the sum could be such that the other the decomposition could be in terms of a plus zero for example
 so what we want to say is that the depth determinant of this is equal to the determinant of a matrix obtained by
 so this is the decomposition that we are talking about plus x y c d ok
 so this is how we have decomposed one determinant into a sum of two determinants just because the elements of one row were able to be decomposable into two elements all right
 so this is what we mean by the statement and how do we show this we can show this again directly by just by the definition again as we have done previously by expanding along the row ah and because now each when you look at the sum each product is going to have a plus x or b plus y or something like that then we just split those and then whatever we are left the two products can go back to its own ah set of determinants
 so let me just write down what i just said ah the idea is as follows
 so if you look at the determinant over here and again like in the last example rather than looking at the formula a plus x times this minus c times b plus y let us say that we are expanding along this row and we have a plus x times its cofactor plus b plus y times its cofactor we know actually that this is nothing but d and this is nothing but minus c but in general we do not know and that ah
 so in general we do not really need to know ah what those are because we are just expanding and we are concentrating along these terms
 so this can be written as a times d plus b times minus c plus

so this is one set of terms plus x times d plus y times minus c now if you look at these this is nothing but the first is nothing but the determinant of this term and the second is nothing but the determinant of this term

so this is a sum of two determinants which are determinant of matrix a b c d plus determinant of the matrix x y c d right

so we have expressed one determinant of the sum of two determinants that is what is meant by this meaning sum of determinants all right

so this is this property which ah to state again says that if you can decompose the sums of a row here we have considered but also of a column in terms of a sum of two terms then the determinant can be expressed as the sum of two determinants ok ah and then the next property which is the last in the series that we are going to talk about is in some sense the opposite of this and what that says is that if you look at a matrix which has a certain determinant and if you add two rows of the matrix and replace one of the rows with them then the determinant is going to be unchanged and that uses to show that would be saying that we use this property and also the property that we had seen earlier about two identical rows meaning that the determinant is zero and whatever we say about the rows also holds for the columns

so let us just see what that property is

so this is property nine and the property nine is that if we have if each element of a row or column is replaced by a sum of that element and an element of another row

so the idea is that each row which already will be replaced by a sum of that element and the element of other rows

so the same other row is considered or column then the value of the determinant remains the same

so let us look at a two by two example to illustrate this suppose we have a matrix a b c d and we replace the first row r1 with r1 plus r2 then this is going to give a matrix a

so we are replacing a with a plus c and b with b plus d and c and d

so another question is what is the determinant of this and what is the determinant of this

so determinant over here is a d minus b c determinant over here is how do we find the determinant of that well just like we saw in the previous property the determinant of this matrix can be replaced as the determinant of the matrix a b c d right because this are two ah sums

so we have plus determinant of c d c d right

so this is from the previous property and ah if you look at just this matrix we see that these two rows are identical and

so this determinant goes to 0 and what we are left with is nothing but determinant of a b c d which is the same as over here and this we see for a two by two matrix but holds true in general for a general n bind matrix

so with this we have finished the nine properties of determinants that we wanted to ah explain for each of them we have looked at some examples what we will do is to look at some problems which use this property in next set of lecture but the idea for looking at these properties are that ok we are going to have to evaluate some determinants ah we could use directly the definition but what we are doing here is by noting some simple determinant calculations and some properties we can manipulate determinants of relatively complicated matrices ah in a simple fashion

so we could obtain the determinants of relatively complicated matrices simplify them and this simplifying idea is behind ah the goal of looking at these properties

so with that i thank you for your attention and i hope that these properties

add a different layer perspective to your understanding of determinants thank you you