

okay hello and welcome to ah these lectures on determinants determinants what determinants are is that they are useful numbers associated with matrices and the goal of these lectures is to introduce what these numbers are where they are used and of course give some idea of the properties

so that they can be properly evaluated

so let me just write down here

so ah it is in particular the square matrices that we are going to be talking about because these numbers the determinants are associated with square matrices and these are some things which we have probably encountered in many contexts and that is where this usefulness comes

so let me just look at some examples where these might have been encountered

so the first example that i want to look at is linear system of equations we can have ah one system of equations as  $x$  plus  $y$  equal to ten and  $4x$  equal to  $y$

so these again come in many contexts at one level one can think of them as being equations of two lines and what we want to do is to solve for their points of intersection

so this can be

so these are just some illustrations of what these equations may represent from a geometric point of view and what we want to see is ok how do we know what are their points of intersection whether there is a point of intersection or not do they intersect at a point

so this is one area as we will see where what we want to define as a determinant is going to play a role um of course these equations can also come up in different areas for example if you talk about as a school algebra problem we have some goal to buy some apples and oranges with the constraint that the number of apples and oranges is equal to ten and that we want to have four times as many apples as oranges

so one could ah compress that information algebraically into these two equations ah saying that  $x$  plus  $y$  is ten and that  $4x$  equal to  $y$  and then you want to find the values of  $x$  and  $y$

so here the goal is to solve the system of equations ok

so how do we do this well there are different ways

so let us try writing them down in a more general format

so these are the two equations rewritten in ah format which suggests that they can be written in a form of a matrix right

so these are just a matrix ah system of equations um one can imagine that we are looking at a two dimensional example but what we want to do in general is going to also work for a  $n$  dimensional system of equations ah more generally we can talk about this matrix as something like  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  where this is the two by two square matrix a general form of a two by two square matrix which comes naturally in these settings

so how do we solve these equations where do the determinants come up let us try to see that one method of solution is as follows we can ah multiply the top equation by  $d$  and the bottom one by  $b$  and subtract the two then what do we get we get that  $a d$  minus  $b c$  times  $x$  is  $d m$  minus  $b n$  okay and in case  $a d$  minus  $b c$  that quantity is not zero then we could divide both sides by that number if  $a d$  minus  $b c$  is not equal to  $0$  then  $x$  is  $\frac{d m - b n}{a d - b c}$

so see

so the in a general setting these are the solution for  $x$  similarly we can come up with the solution of  $y$  in that we write down the equations again multiply the top one by  $c$  bottom one by  $a$  and subtract ok and then writing down this is  $y$  equal to  $\frac{c m - a n}{b c - a d}$  again this is ah for the case where this number  $a d$  minus  $b c$  is not zero

so this is for  $a d - b c \neq 0$  because you know division by 0 is has lot of issues not defined for 1 and what we want to say again going back to the thing is that if we have a matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  a two by two square matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  as  $m \times n$  then solutions found when  $a d - b c$  is not zero now this quantity  $a d - b c$  is nothing but the determinant of this two by two matrix

so this is the determinant of the two by two matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

so this number which naturally comes up in the context of solving a linear set of equations we have looked at a two dimensional example it could be a three dimensional four dimensional  $n$  dimensional example but this is this condition which is nothing but the determinant of the matrix which is important or which is useful in checking whether the system of equations has a solution or not

so the problem of existence of solutions can be found by checking the determinant let me just write that down

so determinant provides a way to check for the existence of solutions in a linear system of equations ok

so through this example i mean this is sort of the objective of this example which was to show how a determinant can be a useful way to check for the existence of a solution in the linear system of equations perhaps that is the reason why the word determinant determinant comes i think the root of this word is the verb determine and that is that it allows us to determine by some calculation which is nothing but the computation of the determinant ah whether the solutions exist or not ah of course these are just some initial examples what we are going to do is formally define a determinant see how they can calculate but here is just a flavor of these interesting combination of numbers or combination of entries of a matrix which come which come up in many contexts and

so the next context which i want to give has a bit more of a geometrical flavor in terms of the area ah and there again what we will see is that the determinant plays an important role

so next example is determinant as an area

so area of what

so the area is of a parallelogram which depends on the entries of the matrix

so again let us consider again here a two by two matrix that we are looking at and let us try to see how it is related to the area of the columns

so let us consider these as vectors

so in the general cartesian coordinate frame if i draw these vectors if i draw one ah particular illustration of these vectors let us say this is the vector corresponding to point  $a$   $c$  and this is the vector corresponding to the point  $b$   $d$  ok

so these are these two vectors and now ah one can think of the parallelogram formed from these vectors as follows

so this and this point being here as  $a + b$   $c + d$  what i want to state what i submit is that the area of this parallelogram is nothing but the same quantity that we were checking in the previous example as being non zero for the existence of solutions and that is what we should formally define later as the determinant

so area is nothing but  $a d - b c$  how does that come

so let us look at that

so let me just draw a bigger version of the same figure a vector here which is  $a$   $c$  a vector here which is  $b$   $d$  and we are trying to enclose it formulate a parallelogram based on those vectors this point being the sum  $a + b$  and  $c + d$  now we want to find out this area  $\Delta$  and the way we would the way we would like to solve for this is to try to find a bigger area and subtract out the different elements what i mean is that we can have this dashed area and

subtract out from this area the area of this triangle the area of this rectangle the area of this triangle the area of this triangle the area of this triangle and then the area of this rectangle

so what is the overall area of the rectangle the big rectangle this is nothing but the length times the breadth which is  $a + b$  times  $c + d$  ok and from here now we have to calculate the individual area

so remember that the area of a triangle is half the base and the height and the area of rectangle as you have just used is the length times the breadth

so what is the area of this rectangle sorry this triangle it is minus half times the base which is nothing but  $a$  and then the height which is  $c$  the area of this rectangle and the height is  $c$  and then the length is  $b - v$   $c$  this triangle the height is again  $d$  and then the base is  $b$

so minus half  $b d$  now we move on to this side the area of this triangle is again minus half the base here is  $b$  and the height is  $d$  minus the area of this rectangle is  $b$  and then the height is  $c$  and the area of this triangle again the base being here is  $a$  and the height is  $c$  minus half  $a c$

so let us just ah this expression simplify it

so delta is  $a + b$  into  $c + d$  and collecting those expressions we have minus  $ac$  minus  $bd$  minus  $2 bc$

so let's simplify this this is  $ac + ad + bc + bd$  minus  $ac$  minus  $bd$  minus  $2 bc$

so this  $bd$  and this  $v d$  cancel one  $a c$  one  $a c$  cancel and one of these  $b c$  is cancel leaving us with  $a d$  minus  $b c$

so this  $a d$  minus  $b c$  term again comes up you know earlier it came up in the context of solving a linear system of equation a very algebraic context here it has come up in a totally different although related context of trying to find the area of a parallelogram made by the columns of a matrix in other words here we have said ok we have  $a b c i$  think it was the other way round this was  $a b$  this was  $c$  this was  $d$  and from here we went to this geometric idea of a parallelogram and this area this delta we have shown is nothing but  $a d$  minus  $b c$

so this number which has now come up in the second context this is what we are going to call as a determinant

so this is what goes back to the initial point that we wanted to understand is that the determinant is a number and not just any number its a useful number why is it useful because it is helpful in finding out the the existence of solutions for a linear system equations it is useful in finding out the area of a parallelogram it has geometric definition and there are many such applications of determinant of a matrix which come up

so to summarize this i just write down that a determinant gives the area of parallelogram formed from the vectors the column from from the columns of the matrix of a matrix  $u_m$  is to be mathematically precise i will just put this area in quotes because this quantity  $a d$  minus  $b c$  can be positive or negative depending on the particular values area is something we usually think of a positive

so i mean the relation still holds we may have to think about the absolute value of the determinant being the actual area although the sign of the area that we get from these computations may also have some other relevance but at our level at this scope let us just take it that the area of this parallelogram is going to be given by the absolute value of the determinant ah again as in before the system of equations was not necessarily constrained to be two dimensional although in the example for purposes of illustration we took a two dimensional system similarly here we don't need to just restrict ourselves to the area we could think of a three by three square matrix and in that case it is

the volume which is going to be the what the determinant or the absolute value of the determinant gives and we could also think about in the higher dimensional spaces ok

so um just for the sake of completeness let us go back to the initial example that we started with just as a example of computing the determinant this was  $\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}$  ok

so this was sort of the 2 by 2 matrix that we generalized into an  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$  matrix and we did two things

so let us see whether this system of equations has a solution and the way to do that would be to calculate the determinant of this matrix

so what is the determinant of this matrix

so  $ad - bc$

so  $1 \times (-1) - 4 \times 1$  which is  $1 - 4$

so this is  $-1 - 4$  is  $-5$

so that is not equal to zero therefore the solution exists again this is a simple example

so we may think ok we could have directly calculated it but we may imagine that for higher dimensions it may be difficult to explicitly solve for the equations and in that case the determinant provides a useful way a quick way to check for the existence of solutions similarly from the point of view of the area also area is let us take the absolute value of the determinant is  $|\det A|$

so this is just taking the absolute value  $|\det A|$  for the reason that i mentioned that you know there is some mathematical meaning to the sign at our level just let us not worry about it too much let us just focus on looking at the number that this gives

so this is sort of completing the circle we started with this example which comes up either in algebraic context or in trying to find the intersectional points of lines and what we see  $|\det A|$  formulating it in terms of matrix problem the solution or the resolution of what we want to do  $|\det A|$  hinges on checking the value of a number or computing this number  $ad - bc$  for a two by two square matrix  $A$  which is nothing but a determinant of it

so therefore this determinant is a useful important number

so going back to what we want to do

so let us say determinants rather than writing this one of the longer words determinants will use notation either  $\det A$  determinant of  $A$  or simply we will just put it in these  $|\det A|$  two vertical lines and we note that this  $A$  is a square matrix and what we want to do is three things first we want to having already seen

so what we have done

so far has just been trying to motivate the need for a determinant in these two  $|\det A|$  different context what we want to do is to first formally define a determinant

so we will define a determinant that is formally come up with a procedure to calculate the determinant for a general  $n$  by  $n$  matrix although we will be using it mostly for two by two or three by three matrices a one by one matrix is essentially just a number

so that determinant is equal to the value of the number after defining a determinant we look at some ways in which one could calculate the determinant

so there may be one method the definition provides one way to calculate the determinant but what we will see are a set of properties which allow us to manipulate the determinant in such a way that it becomes easier to calculate

so that is the other thing that we will see that's the number two properties and finally what we are going to do is to look at some more of the applications of determinants of a similar line to what we just see that you know it helps us to

find the existence of solutions it finds the areas  $a_h$  and some other things

so we look at how what where we can apply the power of this simple number

so this then is the outline of  $a_h$  these set of lectures all right

so um lets start by defining a determinant just to make sure our notations are clear i will start by writing down a general matrix  $a$  and then see  $a_h$  what are the sub elements sub definitions which need to be made before finally coming up with the definition of the determinant

so  $a_h$  is a general square matrix let us say an  $n$  by  $n$  square matrix and also write it in this representation where  $a_{ij}$  represents the  $i$ th row and  $j$ th column

so  $i$  row  $j$ th column or if you try to write down the matrix as a whole this would be something like  $a_{11}$  because is the first row and the first column then  $a_{12}$  first row second column

so on then the first row  $n$ th column next row  $a_{21}$  second row first column  $a_{22}$  second row second column

so on second row  $n$ th column and likewise we'll have the  $n$ th row first column  $n$ th row second column and then the final one  $n$  through  $n$ th column

so this is writing down the matrix in its full detail or we could also just think of it writing the  $i$  throw  $j$ th columns

so column  $j$  row  $i$  and this entry is  $a_{ij}$  right

so this is a general  $n$  by  $n$  matrix we have already seen an example of a 2 by 2 matrix and what we want to do is to say ok how do we now goal is to say ok what is the determinant of  $a$

so goal what is the determinant of  $a$  in terms of  $a_{ij}$

so for that what we want to do first is we have to define what is called as a minor

so a minor

so minor is a quantity which is associated with each entry

so minor  $m_{ij}$  is associated with each entry okay  $a_h$  and how is the minor  $a_h$  defined it is defined by  $a_h$  determinant of the matrix obtained by deleting the  $i$ th height row and the  $j$ th column

so if you have a matrix which has entries  $a_{ij}$  if you delete the  $j$ th column and the  $i$ th row we are remaining by another square matrix which is now of dimension  $n$  minus 1 minor is nothing but the determinant of that matrix it is the determinant of a matrix obtained from  $a$  after deleting the  $i$  throw and  $j$ th column for example let us take an example again this matrix that we are familiar with suppose  $a$  was a two by two matrix that we had encountered previously one one four minus one ok

so so this this is a one 1 this is a 1 2 this is a 2 1 and this is a 2 2

so what is the minor of a one two  $i$  that is  $m_{12}$  one two  $m_{12}$  is nothing but the determinant of a matrix obtained by deleting the  $a_h$  first row and the second column

so we will delete the first row this one and then we will delete this second column and

so  $m_{12}$  is nothing but the determinant of the matrix which has a single element four which as we have said before is nothing but four

so we are we are going to say that the determinant of a scalar one scalar number is always a the same scalar

so the minor is 4 as we take determinant of scalar as itself okay

so this is  $a_h$  the definition of a minor and then closely related to the definition of a minor is this idea of a cofactor

so cofactor just like the minor it is also associated with each element of a matrix

so we call it denote it as  $a_{ij}$  and the co factor is equal in magnitude to the

minor but it can have a different sign and the sign will depend on the value of the row  $i$  and the value of the column  $j$  index

so this cofactor is defined as  $a_{ij}$  is minus 1 power  $i + j$  into  $m_{ij}$   
so it is this power of minus 1 depending on whether  $i + j$  is even or odd which will determine whether the cofactor is equal to the minor or it is equal to minus times the minor

so in the previous example what would be the cofactor  $a_{12}$  is going to be minus 1 power  $1 + 2$  into  $m_{12}$  which is  $a_{12}$  minus one cube  $m_{12}$  and minus one cube is minus one

so this is minus  $m_{12}$  and since  $m_{12}$  was four this is minus four  
so the cofactor this is an example of a cofactor calculation and we will do some more examples of this  $a_{ij}$  to define to say what

so eventually we want to define a determinant which is the next step but before that we need these sub elements of minor and cofactor now we may think ok why do we need to define

so many things already we know for a two by two matrix with  $a$   $d$  minus  $b$   $c$  for a matrix  $a$   $b$   $c$   $d$  for a general three by three for a general  $n$  by  $n$  why don't we define it like that the reason is that we want like a simple scalable way to define a determinant for elegance with the elegant notation  $a_{ij}$  which for it to be easy to represent in a simplified manner

so with these definitions of the minor and the cofactor we can now define the determinant

so determinant is the sum of the product of elements of a row or a column with their cofactors ok

so what we mean here let us just go through this  $a_{ij}$  what we are saying is that the determinant is the sum of a product  $a_{ij}$  and the sum contains basically the sum contains  $a_{ij}$  individual elements which are products what are these products these are products of the elements of a matrix

so one row or one column and they are products of that element with their corresponding cofactors

so the corresponding cofactors

so mathematically we would say that the determinant of  $a$  is  $a_{ij}$  the summation over  $i$  of  $a_{ij}$  times  $a_{ij}$  for a fixed

so we may either fix a column and sum over the rows or we could fix a row and sum over the column any column

so all this is equal to the summation of  $j$   $a_{ij}$   $a_{ij}$  for fixed  $i$

so that is how we define a determinant ok  $a_{ij}$

so many things we should we have possibilities to do i think the one thing or the first thing that we should do is to go back and check whether for a two by two matrix  $a$   $b$   $c$   $d$  the determinant that we get from this definition is whether it is equal to  $a$   $d$  minus  $b$   $c$  or not whether we can derive that  $a_{ij}$  then how do we come up with a determinant for a higher dimensional matrix that is the other thing that we need to see and of course the thing that we have taken is that the determinant of a scalar which is nothing but a one by one matrix is itself

so so the examples that we can consider now are

so one by one one by one determinant  $a_{ij}$

so  $a_{ij}$  apologies

so one by one matrix which is a scalar its determinant was taken as the scalar itself ok

so one by one matrix anyway is a scalar each determinant is a scalar itself  $a_{ij}$  next two by two matrix let see what it comes out to

so we have already worked with a 2 by 2 matrix but now let us do it in a formal rigorous setting

so here let us start by expanding along this row ok

so what we need to do is to write down this as  $a$  times the cofactor of  $a$   
so what is the cofactor of  $a$  cofactor of the element  $a$  is nothing but the  
matrix or the determinant of the matrix which comes after deleting the row and  
the column of which  $a$  is part of

so essentially just the element  $d$  because if you cancel this row and this  
column we are left with  $d$  but times minus 1 power the sum of the row index and  
column index of  $a$

so the row index of  $a$  is one and the column index of  $a$  is also one

so this is going to be minus one one plus one

so minus one square

so this is nothing but  $d$  itself

so this is over here  $d$  ah similarly when we expand in terms of this row

so we have  $a$   $d$  and then the next term in the sum is the product of  $b$  the  
element  $v$  and its corresponding cofactor

so this is plus  $b$  times the cofactor of  $b$

so what is the cofactor of  $b$  cofactor of  $b$  is the determinant of the matrix  
which comes after deleting the row and the column of that  $b$  belongs to and

so that is nothing but  $c$  but it is multiplied by minus one power the sum of the  
row and column index of  $b$

so if you see  $b$  belongs to the first row but second column

so this is minus 1 power 1 plus 2

so minus 1 cube

so it is minus  $c$

so this number goes over here

so all in all the determinant determinant of this matrix is going to be  $a$   $d$   
minus  $bc$  because it is minus  $c$  which comes over here

so again we are go back to our expression which we have seen multiple times  
which is  $a$   $d$  minus  $b$   $c$  we could also ah expand along

so here we expanded along this row we could check by expanding along this row  
or this column or this column what happens

so lets do one of them lets expand along another column ah the reason for doing  
this is because the determinant definition the way we have said it is that it  
can either expand along a row in fact any row but it has to be one row ah and it  
can also expand around a column or any column it but it has to be one column

so let us try to check that by expanding along a different column and see  
whether we get recover the same number or not

so here is the matrix  $a$   $b$   $c$   $d$  and let us expand along this as before what we  
have to do is to say ok it is a sum of a product the product first product term  
is going to be  $b$  times its cofactor ah plus  $a$  times its cofactor

so again what is the cofactor of  $b$  that does not change whether we expand along  
a row or a column that we already calculated previously to be minus  $c$  ok ah and

so since we are expanding along this column this entry should actually be  $d$  and  
not  $a$  and what we need to multiply with this  $d$  is its corresponding cofactor

so we have not calculated the cofactor of  $d$  before in the previous page we  
calculated the cofactor of  $a$  and  $b$  but now i think for a two by two matrix it is  
relatively straight forward to calculate the cofactor

so just imagine ah blanking out the column that  $d$  belongs to

so no  $b$  and no  $d$  and also the row that  $d$  belongs to

so no  $c$

so what we are left with is just  $a$

so that is what we are multiplying  $d$  by and as we see this wonderful expression  
comes out to be again  $a$   $d$  minus  $b$   $c$  which is the same as before

so this is a good sanity check a good consistency check that we see

so i think um if nothing else one of the things which should stay back from

this lecture is just this quantity  $a d - b c$  which is related to a square matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

so what we have seen is how  $\det A$  comes up through the formal definition of the matrix determinant and also even if you do not consider the determinant definition explicitly when we try to solve like a two dimensional system of equations and again it is the determinant of the associated matrix which tells us whether the solutions may exist or not  $\det A$  or if you look at geometrically the area of a parallelogram formed by the columns of the matrix then this number tells us the area

so this is in a sense a very magical number to have

so we have seen this in multiple contexts we have seen this as formal definition of determinant we have seen this as a way number to check existence of solutions and we have also seen this number as area of parallelogram all right um the next thing to do is to now look at a three by three matrix in principle we have already defined the determinant for a general matrix

environmentic  
so in a sense we should already know how to calculate it but there is a lot of merit in actually looking at specific examples and making the calculation

so let us look at a three by three matrix and in this case let us take a numerical example

so the 3 by 3 matrix that we have here has the entries  $\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \\ 5 & 2 & 0 \end{pmatrix}$

so the question here is what is the determinant

so how will we calculate the determinant what we have to do is pick any row or column looking at the matrix um you know for reasons of calculations we always want to do the less amount of work or the more efficient amount of work

so as you see here there are three rows three columns in principle nine choices but one row and one column have  $\det A$  one zero in them

so that automatically tells us that the entry is zero

so you can multiply it with whatever is its cofactor that term is going to be zero

so we do not need to calculate the cofactor

so we can get away by calculating just two cofactors if you expand either along the first row or the third column

so let us now expand along the first row

so we have one times the cofactor

so cofactor is going to be  $-1$  times the determinant of the matrix obtained by deleting the first row and the first column

so that is  $-1 \times \det \begin{pmatrix} 2 & 2 \\ 5 & 0 \end{pmatrix}$  then there is going to be the third term the second term in the sum apologies is going to be zero now we do not really care what the cofactor is because it is multiplied by zero

so it is going to be zero and then the third term is going to be two which is this entry we are expanding along this column this row apologies

so this is going to be  $2 \times \det \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$  times the determinant obtained by deleting the first row and this column

so  $3 \times \det \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix}$ .

so this term is already 0

so this determinant is going to be  $\det A = 1 \times (-1) \times \det \begin{pmatrix} 2 & 2 \\ 5 & 0 \end{pmatrix}$

so  $1 \times \det \begin{pmatrix} 2 & 2 \\ 5 & 0 \end{pmatrix}$  now we know we have done this 2 by 2 deep determinant calculations many times

so now we can just say ok this is  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

so determinant of this is going to be  $a d - b c$

so minus 1 into 0 minus 2 into 2 times here  
so plus 2 and minus 1 power 4 is 1 again a 2 by two determinant  
so we say three into two minus minus one into five  
so this is zero  
so here we get minus four plus two times ah 6 and then this is 5  
so this is 11 11 times 2 22 minus 4  
so minus 4 plus 22 which is equal to 80.

so this is the value or how we can compute the determinant for a numerical example all right

so this is a good point to summarize what we have done in this lecture  
so we have talked about determinants all right and we talked about defining a determinant

so in this we ah start by noting the determinant of a scalar is the scalar itself and then from there move on to defining a two by two three by three and a general n by n determinant through the ah calculation of the element the elements minors and cofactors ah and before that we looked at some places where these determinants arise

so these arise in many contexts such as ah consistency or finding out whether solutions exist exists and in computing areas i mean these are some places where they arise and i think historically these have been used for these and perhaps many other contexts almost since the 1600s or

so ah you can imagine that the ideas of calculating the areas in finding out whether there are systems of linear equations what are the solutions whether they exist or not like finding out the intersections of lines these have problems which have been thought about for many years and what is very interesting is that the calculation of these determinants the use of these determinants continues to the present edge ah there are many many areas in engineering in science where the idea of calculating the determinants and then the higher level advanced concepts around the calculation of determinants are very useful

so ah that is what we hope to cover in the next lecture is to look at some properties of the determinants which enable many of these applications

so next we look at the properties in their evaluation

so we have to look at properties or determinants which will help in the evaluation

so this is sort of the program for the next time we look at the properties of the determinants and thank you for your attention and look forward to look at the other aspects of the determinants in the coming lectures you