

hello students welcome to iit palm mathematics problem solving session this is lecture number five in this lecture i will solve some more interesting problems related to system of linear equations

so let's start with problem number one question let  $s$  be in the set of all column matrices  $b_1 \ b_2 \ b_3$  such that  $b_1 \ b_2 \ b_3$  they are real numbers and the system of equations in real variables  $x + 2y + 5z = b_1$ ,  $2x - 4y + 3z = b_2$ ,  $x - 2y + 5z = b_3$  has at least one solution okay then which of the following systems in real variables have at least one solution for each  $b_1 \ b_2 \ b_3$  belongs to  $s$  the first system is  $x + 2y + 3z = b_1$  second equation is  $4y + 5z = b_2$  third is  $x + 2y + 6z = b_3$  second part is okay  $x + y + 3z = b_1$ ,  $5x + 2y + 6z = b_2$  and  $-2x - y - 3z = b_3$  part c is  $-x + 2y - 5z = b_1$ ,  $2x - 4y + 10z = b_2$  and the third equation is  $x - 2y + 5z = b_3$  part d is  $x + 2y + 5z = b_1$ ,  $2x + 3z = b_2$ ,  $x + 4y - 5z = b_3$  okay

so let's go back and read this question carefully

so here  $s$  is the set of all column matrices  $b_1 \ b_2 \ b_3$  says that the following system of equation has at least one solution

so then which of the following system will has at least one solution for each  $b_1 \ b_2 \ b_3$  okay

so first of all first of all we need to find the set  $s$  right

so what is the condition condition is that this system has at least one solution right

so  $s$  is the set of all those  $b_1 \ b_2 \ b_3$  for which the system has at least one solution okay

so let's uh let's form first from the set yes okay okay

so let's solve this problem

so consider the system of indication which is given in the question okay  $x + 2y + 5z = b_1$ ,  $2x - 4y + 3z = b_2$ ,  $x - 2y + 5z = b_3$

so right write the augmented matrix for this system

so which is nothing but  $\begin{bmatrix} 1 & 2 & 5 & b_1 \\ 2 & -4 & 3 & b_2 \\ 1 & -2 & 5 & b_3 \end{bmatrix}$  okay

so now let's uh let's apply some row operation for example let's apply this transformation  $r_2 \leftarrow r_2 + 2r_1$  and  $r_3 \leftarrow r_3 + r_1$  okay

so under this transformation

so let's see how what is how this matrix is reduced okay let's write it in the next page okay

so the first row there is no change  $\begin{bmatrix} 1 & 2 & 5 & b_1 \\ 0 & 0 & 13 & b_2 + 2b_1 \\ 0 & 0 & 10 & b_3 + b_1 \end{bmatrix}$  now  $r_2$  is replaced with the  $r_2 + 2r_1$

so this is  $0$  and this is  $0$ ,  $3 + 10 = 13$ ,  $b_2 + 2b_1$  okay now this one is  $r_3 + r_1$

so  $0$  and  $0$  and it's  $7$  and  $b_3 + b_1$  okay now we need to reduce it further because

so far we cannot say anything about the rank of these matrices ok

so now i will apply this  $r_3 \leftarrow r_3 - 7r_2$

so  $\begin{bmatrix} 1 & 2 & 5 & b_1 \\ 0 & 0 & 13 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 7(b_2 + 2b_1) \end{bmatrix}$  no change now here  $0 \ 0$  this is  $0$  okay this is  $0$  and then  $b_3 + b_1 - 7(b_2 + 2b_1)$  okay

so let me erase this okay yeah  $7$  by  $13$   $r_2$  it's a  $v_2 + 2$  times  $b_1$  all right okay

so now what do we get we get minus

so this is equals to  $\begin{bmatrix} 1 & 2 & 5 & b_1 \\ 0 & 0 & 13 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 7(b_2 + 2b_1) \end{bmatrix}$  and this is nothing but  $13 \ b_1$  uh  $-13 \ b_1 \ b_3 + 13 \ b_1 - 7 \ b_2 - 14 \ b_1$  divided by  $13$

so we get  $6 - b_1 + 13b_3$  okay let's let's write it in the next  
 so we get the following system  $\begin{bmatrix} 1 & 2 & 5 \\ v & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 13b_2 + b_1 \\ 0 \\ 0 \end{bmatrix}$  and  
 here we get  $-b_1$   
 so  $-b_1 - 7b_2 + 13b_3$  divided by 13  
 so so given that the system has at least one solution therefore  $-b_1 - 7b_2 + 13b_3$  has to be 0 all right then only the rank of augmented matrix  
 will be same as the rank of a because here the rank of a is 2 and but we cannot  
 say anything about the rank of a appointed b  
 so if this  $-b_1 - 7b_2 + 13b_3 = 0$  then the rank of augmented  
 matrix will also be 2 okay  
 so this implies that  $13b_3$  is equals to  $b_1 + 7b_2$   
 so this is the condition  
 so s set is the set of column matrix  $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$  belongs to  $\mathbb{R}^3$  such that  $13b_3$   
 which equals to  $b_1 + 7b_2$  okay  
 so that we have formed the s set  
 so now let's take the part a and see whether that system has at least one  
 solution for all  $b_1, b_2, b_3$  from s okay  
 so the system is this  $x + 2i + 3z$  is equals to  $b_1$   $4y + 5z$  is equals to  
 $b_2$  and  $x + 2i + 6z$  is equals to  $b_3$   
 so let's call the system one okay  
 so write the augmented matrix of the system  $\begin{bmatrix} 1 & 0 & 3 & b_1 \\ 0 & 4 & 5 & b_2 \\ 1 & 2 & 6 & b_3 \end{bmatrix}$  okay  
 so let's reduce this let's reduce this  
 so what do we do we will apply the transformation  $r_3 - r_1$   
 so we apply the  $r_3 - r_1$  and goes to  $r_3 - r_1$  okay  
 so the first row is as it is one two three  $v_1$  even second row is also  $0 \ 4 \ 5 \ b_2$   
 and this is  $0 \ 0 \ 3 \ b_3 - b_1$   
 so here rank of a augmented b is same as the rank of a which is equals to 3  
 right this implies that the system because we already get the three non-zero  
 rows and then the this matrix is also in the it's long form this implies system  
 one has unique solution for all  $b_1, b_2, b_3$  okay therefore in particular therefore  
 in particular one has system of system one has at least one solution for all  $b_1, b_2, b_3$   
 belongs to s okay  
 so let's go to the second part b which is nothing but  $x + y + 3z$  is  
 equals to  $b_1$   $5x + 2y + 6z$  is equals to  $b_2$   $-2x - y - 3z$  is  
 equals to  $b_3$  all right  
 so let's check whether this system has at least one solution for all  $b_1, b_2, b_3$   
 belonging to s let's solve the system two okay  
 so write the augmented matrix for the system  $\begin{bmatrix} 1 & 1 & 3 & b_1 \\ 5 & 2 & 6 & b_2 \\ -2 & -1 & -3 & b_3 \end{bmatrix}$  okay  
 so this okay  
 so what we'll do is we'll do some row transformation  
 so first one is this  $b_1$  the transformation is  $r_2 - 5r_1$   
 so this is  $0 \ 2 \ -10 \ 2b_2 - 5b_1$   $2 \ -5 \ 2 \ -5b_2 + 10b_1$   $3 \ -6 \ -15 \ -3b_2 + 6b_1$   
 $2 \ -5 \ 2 \ -5b_2 + 10b_1$   $v_1$  okay let me just write  $r_2 - 5r_1$  similarly i'll  
 take this operation  $r_3 + 2r_1$   $r_3 + 2r_1$  sorry we don't have a space  
 here ok  $r_3 + 2r_1$   
 so this becomes  $0$  then  $-1 + 2$  becomes  $1$   $-3 + 6$  becomes  $3$  and  $b_3 + 2b_1$   
 $3 + 2b_1$  okay  
 so now we need to we need to make these entries 0 all right  
 so in the next time what we will do is we'll apply the transformation  $r_3$  which  
 is replaced by  $r_3 + \frac{1}{3}r_2$  okay  
 so let's see with this transformation  $r_3$  elementary row operation  $r_3 + \frac{1}{3}r_2$   
 $1 \ 3 \ r_2$

so this will reduce the system of equation further

so  $1 \ 1 \ 3 \ b_1$  there is no change in the second row  $0 \ -3 \ -9$  and  $b_2$  minus  $5 \ b_1$  and here if i multiply the row 2 by  $\frac{1}{3}$  we get  $1 \ -1$  which is and add it to the third one it's a zero and three minus three is zero okay and we have  $b_3$  plus two  $b_1$  plus one by  $3 \ b_2$  minus  $5 \ b_1$  okay

so what do we get we get  $1 \ 1 \ 3 \ 0 \ -3 \ -9 \ 0 \ 0 \ 0$  this is  $b_1$  this is  $b_2$  minus  $5 \ b_1$  and this is  $6 \ -5$

so  $b_1$  plus  $b_2$  plus  $3 \ b_3$  divided by  $3$

so this is the reduced system we get okay

so this system will have a solution only if  $b_1$  plus  $b_2$  plus  $3 \ b_3$  is equals to  $0$ .

so this implies that for system two to have at least solution we should have one solution for each  $b_1 \ b_2 \ b_3$  belonging to this we should have  $b_1$  plus  $2 \ b_2$  plus  $3 \ b_3 = 0$  for all  $b_1 \ b_2 \ b_3$  from  $S$  because then only the rank of augmented matrix will be same as the rank of  $A$  because rank of  $A$  here if you see it's a  $2$  and if  $b_1$  plus  $b_2$  plus  $3 \ b_3$  is  $0$  then then the rank of augmented matrix will also be two and then there will be at least one solution okay

so so that

so that means this condition should be true okay

so but this condition does not hold is not true for all  $b_1 \ b_2 \ b_3$  belonging to  $S$  because for example  $(6 \ 1 \ 1)$  belongs to  $S$  because to be in  $S$  we should have that  $13 \ b_3$  is equals to  $b_1$  plus  $7 \ b_2$

so this is true right what is the condition for being in  $S$

so  $13 \ b_3$  is equal to  $b_1$  plus  $7 \ b_2$ .

so definitely this this is true right this point belongs to this but let's call this condition three condition three it does not hold

so this means the system two does not have at least one solution will not have at least one solution for each  $b_1 \ b_2 \ b_3$  belonging to  $S$  okay

so let's go to the part c which is system is  $x$  plus  $2y$  minus  $5z$  is equals to  $b_1$   $2x$  minus  $4y$  plus  $10z$  is equals to  $b_2$   $x$  minus  $2y$  plus  $5z$  is equals to  $b_3$  okay

so let's call this system sorry install the system three you know three values

so four okay let me raise this okay

so let's consider the augmented matrix for the system  $1 \ 2 \ -5 \ 2 \ -1 \ 4 \ 10 \ 1$  minus  $2 \ 5 \ b_1 \ b_2 \ b_3$

so let's reduce this

so we'll just take minus  $1$  to minus  $5$  and  $b_1$  we apply that  $r_2$  goes to  $r_2$  plus  $2 \ r_1$   $r_3$  plus  $2 \ r_1$  we get  $0$  and here also we get  $0$  and here also we get  $0$  and then  $r_3$  plus  $r_1$  we get  $0 \ 0 \ 0$  and here  $b_2$  plus  $2b_1$  and  $b_3$  plus  $b_1$

so what is the row operation we used we used  $r_2$  plus  $2r_1$  and  $r_3$  is  $r_3$  plus  $r_1$ .

so here this implies what this implies rank of  $A$  is one

so for this system for to have at least one solution we need both  $b_2$  plus  $2 \ b_1$  is  $0$  and  $b_3$  plus  $b_1$  is equal to  $0$

so both the condition should be should be true but  $(6 \ 1 \ 1)$  as we discussed in the last part belongs to  $S$  it does not satisfy both these transitions ok therefore system 4 will not have at least will not have at least one solution for all  $b_1 \ b_2 \ b_3$  belonging to  $S$  okay

so let's go to the last part which is d and here the system is given by  $x$  plus  $2y$  plus  $5z$  is equal to  $b_1$   $2x$  plus  $3z$  is equals to  $b_2$   $x$  plus  $4y$  plus  $1$  plus minus  $5z$  is equal to  $b_3$

so we need to check whether this has an this has at least one solution for all

$b_1, b_2, b_3$  for all  $b_1, b_2, b_3$  which belong to  $S$ . Let's call this system five okay

so let's write the augmented matrix of the system  $\begin{pmatrix} 1 & 2 & 5 & b_1 & b_2 & b_3 \\ 0 & 4 & -5 & b_1 & b_2 & b_3 \end{pmatrix}$  and then you just reduce it

so what we do is  $R_2 - 2R_1$  then we substitute  $R_2$  minus we apply  $R_2 - 2R_1$  so this is  $\begin{pmatrix} 1 & 2 & 5 & b_1 & b_2 & b_3 \\ 0 & -4 & -10 & -b_1 & -b_2 & -b_3 \end{pmatrix}$  and then  $R_3 - R_1$

so zero it's uh two and this is minus ten all right and  $b_3 - b_1$  okay

so now next we'll just do the elementary row operation

so here let me just write it what operation we used used at  $R_2 - 2R_1$  and  $R_3 - R_1$  okay

so here we will use the operation  $R_3 + \frac{1}{2}R_2$  and then we get  $\begin{pmatrix} 1 & 2 & 5 & b_1 & b_2 & b_3 \\ 0 & -4 & -10 & -b_1 & -b_2 & -b_3 \\ 0 & -7 & -10 & -b_1 & -b_2 & -b_3 \end{pmatrix}$  okay then  $R_3 - \frac{7}{4}R_2$

so it's  $-10 - 7 \cdot \frac{1}{2} = -10 - 3.5 = -13.5$  okay and then  $b_3 - b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_2$  okay

so anyway this system has at least one solution for all  $b_1, b_2, b_3$  because here we can say that rank of  $A$  is equal to rank of  $[A|b]$  i'm going to double  $b$  which goes to 3 for all  $b_1, b_2, b_3$

so this implies that system 5 has at least one solution for all  $b_1, b_2, b_3$  belonging to  $S$  okay

so that means the system one and system four will have i mean part a and part four system defined in part and part four has at least one solution for all  $b_1, b_2, b_3$  and the point  $b$  parts it doesn't have it okay

so let's solve another problem question let  $x, y, z$  be points with integer coordinates that is  $x, y, z$  are integer satisfying the system of homogeneous equations  $3x - y - z = 0$ ,  $3x + z = 0$ ,  $3x + 2y + z = 0$  then how many such points satisfy  $x^2 + y^2 + z^2 \leq 100$  okay

so let's solve this okay

so here if you see these are the system of homogeneous equations

so homogeneous system of equation has  $b$  which is  $0$

so here  $b$  is  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$   $b$  is  $0$  vector and  $A$  is  $\begin{pmatrix} 3 & -1 & -1 \\ 3 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

so in this case rank of  $A$  is always equals to the rank of augmented matrix  $[A|b]$  the reason is because  $b$  is  $0$  vector all right

so let's just try to find the rank of  $A$

so we have  $A$  which is given by  $\begin{pmatrix} 3 & -1 & -1 \\ 3 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix}$  okay then we just apply the row transformation which is  $R_3 - R_1$  and  $R_2 - R_1$

so the first row is same okay then this is  $\begin{pmatrix} 3 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix}$  this is  $0$  this is  $1$  and this is  $0$  all right

so now let's reduce it further here we will use the  $R_3$  as  $R_3 - 3R_2$  then we will have  $\begin{pmatrix} 3 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$  all right

so here if you see the rank of  $A$  is 2 right rank of  $A$  is equals to 2 which is less than 3 this implies system has infinite infinitely many solutions okay

so so let's take the reduced system of equation

so what is that  $\begin{pmatrix} 3 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$  and  $x, y, z$  is equal to  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  all right and this gives us  $3x - y - z = 0$  and  $y = -2z$  and  $z = -\frac{3}{4}x$  all right

so this implies that if you take  $x$  as a  $\alpha$  then  $\alpha = 3\alpha$  with that and for integer because we are interested in integer solution okay is set of integer solution of system of equation system of equations okay

so we need to find that how many such solutions satisfy the  $x^2 + y^2 + z^2 \leq 100$  okay

so for  $\alpha \in \mathbb{Z}$  to satisfy

so this by  $x^2 + y^2 + z^2 \leq 100$  we have  $\alpha^2 + 9\alpha^2 \leq 100$  this implies that  $10\alpha^2 \leq 100$  this implies  $\alpha^2 \leq 10$  okay given  $\alpha$  is integer okay this implies  $\alpha \in \{-3, -2, -1, 0, 1, 2, 3\}$  all right then the integer solutions

so this is fine  $x^2 + y^2 + z^2 \leq 100$  are given by  $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (3, 0, 0), (0, 3, 0), (0, 0, 3), (-1, 0, 0), (-3, 0, 0), (0, -1, 0), (0, -3, 0), (0, 0, -1), (0, 0, -3)$

so this means total number of points total number of points is 7 okay

so this is the this is the final answer let's solve another question

so consider the following linear equations  $x + by + cz = 0$ ,  $bx + cy + az = 0$ ,  $cx + ay + bz = 0$  then if  $a + b + c \neq 0$  and  $a^2 + b^2 + c^2 = ab + bc + ca$  then show that the equations represent the identical plane by two if  $a + b + c \neq 0$ ,  $a^2 + b^2 + c^2 = ab + bc + ca$  then show that the equation represent planes meeting at a single point okay the third part is if  $a + b + c = 0$  and  $a^2 + b^2 + c^2 = ab + bc + ca$  show that equations represent the whole of  $\mathbb{R}^3$  okay

so let's solve this problem the point one okay

so here  $a + b + c \neq 0$  and  $a^2 + b^2 + c^2 = ab + bc + ca$

so this simply means that  $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

so this implies that  $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$  right okay

so this implies that because  $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

so this is the this is basically this is the sum of sum of three positive uh non-negative numbers and the sum is 0

so this implies that each and individual term has to be 0 this implies that  $a - b = 0$ ,  $b - c = 0$ ,  $c - a = 0$ .

all right

so this implies that  $a = b = c$  and because  $a + b + c \neq 0$

so this is not equal to 0 all right

so we obtain that this  $a = b = c$

so this implies that equations represent the identical plane okay

so let's go to pi 2

so the pi 2 we have  $a + b + c \neq 0$  and  $a^2 + b^2 + c^2 = ab + bc + ca$  all right okay

so here let's let's take this coefficient matrix  $A$  which is nothing but  $\begin{pmatrix} a & b & c \\ bc & ca & ab \end{pmatrix}$  all right

so what is the what is the determinant of  $A$  is nothing but  $a^3 - 3abc$  okay

so this is nothing but  $3abc - a^3 - b^3 - c^3$  all right so as we know that  $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$  all right this is what we know

so this implies that determinant of  $A$  is equals to minus of  $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

so given  $a + b + c$  is non zero and  $a^2 + b^2 + c^2$  also not equal to  $a^2 + b^2 + c^2$  this implies that determinant of  $A$  is zero

so this implies that system of equation not zero sorry system of homogeneous equation homogeneous equation will have a unique solution and which is nothing but  $x$  is equal to zero  $y$  is equals to zero  $z$  is equals to zero

so the trivial solution is the only solution

so this implies that the planes are meeting this implies that equations represent planes meeting at single point which is the zero zero  $\theta$  okay all right

so let's go to third party

so what is the  $a + b + c$  is  $\theta$  and  $a^2 + b^2 + c^2$  is equal to  $a^2 + b^2 + c^2$

so these are the two conditions

so as we did in the first part  $a^2 + b^2 + c^2$  is equals to  $a^2 + b^2 + c^2$  gives  $a = b = c$  okay and this implies that  $a = b = c$  this implies that  $a = b = c$  this implies that  $a = b = c$  is equals to  $\theta$

so this implies that any  $x, y, z$  belongs to  $\mathbb{R}^3$  will be the solution of system of equations okay

so this implies that equation represent the whole space hole of  $\mathbb{R}^3$

so that's it i will i'll stop here okay students i will stop here now this was the last lecture of this series thank you for attending these problem solving sessions i hope you enjoyed them i wish you good luck thank you you