

hello students welcome to iit palm mathematics problem solving session this is lecture number four in today's lecture first i will work out one problem related with matrices then i will start system of linear equations for which i'll give great background and then i will work out some interesting problems based on system of linear equations okay

so let's start with the problem question let p_1 equals $\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$ p_2 is another 3×3 matrix given by $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$ p_3 is $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ p_4 is $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$ p_5 is $\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ and p_6 is $\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$.

so these are the six matrixes

so here if you see in each row and a column there is exactly one one and two zeros okay then let's show the following then the part one is okay let me write it on another page then part is if x okay oh there is there is some more thing in there in the question okay

so we had these six matrices and x is another matrix which is given by k is equals to $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ p_k into $\begin{pmatrix} 2 & 1 & 3 & 1 & 0 & 2 & 3 & 2 & 1 \end{pmatrix}$ p_k transpose okay then the part is if x of $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is equals to α times $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ then α equals to 30 part b is so that x is symmetric matrix for c is x minus 30 i is not an invertible matrix okay

so let's solve this problem okay

so we just denote this matrix b is equals to $\begin{pmatrix} 2 & 1 & 3 & 1 \end{pmatrix}$ let me just first row is $\begin{pmatrix} 2 & 1 & 3 & 1 & 0 & 2 & 3 & 2 & 1 \end{pmatrix}$ okay

so it is clear that b transpose is equals to b

so b is symmetric matrix right okay

so the party

so okay let's solve the party okay

so we denote this matrix will be okay then x is equals to k equals to $\begin{pmatrix} 1 & 2 & 6 \end{pmatrix}$ p_k b p_k transpose okay

so okay

so let's x one one is equals to k equals to $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ b p_k transpose $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ okay

so what is the p_k transpose one one one

so if you look at the all these p_1 transfers p_2 transfer p_3 transpose up to p_6 transpose here in each exact in each row there is exactly one one and two are 0 's

so p_k transpose $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is nothing but $\begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 6 \end{pmatrix}$ p_k b $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ the reason is because p_k transpose $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is equals to $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ ok

so what is the b $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

so b is the matrix which is given above then b $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is nothing but k is equals to $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ p_k and b_{11} is 6 3 and 6 okay okay

so x will be nothing but p_1 plus p_2 plus p_3 plus p_4 plus p_5 plus p_6 and this is $\begin{pmatrix} 6 & 3 & 6 \end{pmatrix}$ okay

so if you add all these matrices we get 2 times $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ and this is $\begin{pmatrix} 6 & 3 & 6 \end{pmatrix}$ okay

so okay yeah this is not actually x this is x times $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ all right we can check in the previous slide yes all right

so what is this this is nothing but 2 times $\begin{pmatrix} 15 & 15 & 15 \end{pmatrix}$ all right this is equals to 30 times $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ okay

so what is the x $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ this implies that x $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is equals to 30 times $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and in the question x $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is given α times $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is equal to 30 times $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and this implies that α minus 30 times $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is equals to 0 this implies α is equal to 30

so this is what we wanted to prove okay

so let's go to the part 2 which part b we need to show need to show x is

symmetric matrix okay

so for that we will take the x transpose

so x transpose is where x is $p \times k$ okay we denoted that matrix by $p \times k$ b $p \times k$ transpose k is equals to $1 \times 2 \times 6$ and this is the transpose

so what is this this is nothing but k is equals to $1 \times 2 \times 6$ $p \times k$ transpose b transpose $p \times k$ no $p \times k$ b transpose $p \times k$ transpose all right and given b is a symmetric matrix this is what we have seen in the part a

so this is equals to $6 \times p \times k$ b $p \times k$ transpose and this is nothing but x

so this implies that x is a symmetric matrix okay

so let's go to the part c part c we need to show that x minus 30 i is not an invertible matrix ok

so from part a we have x one one one is equals to thirty times one one one okay

so this is nothing but x minus 30 i $1 \times 1 \times 1$ is equals to 0 .

okay this implies one one one is non-trivial solution of x minus 30 i into y is equals to 0 okay

so this implies that x minus 30 i is invertible is not invertible because if x minus 30 i is invertible then x minus 30 i y equals to 0 has only 0 solution but in this case we have we have a non-zero solution which is $1 \times 1 \times 1$ okay

so this is the reason that x minus 30 i is not an invertible matrix okay okay students

so let's start a new topic which is a system of linear equations i'll give brief background on this topic which will help you in solving the problems related with the system of linear equations

so let's start with the background system of linear equation

so before solving the problems related to the system of linear equation i'll give a brief background on this topic

so let a $n \times n$ cross n matrix x b and n cross 1 vector and b n and n cross one vector okay then system of an equation linear equations in variable x can be written as ax is equals to b

so any x which satisfies 1 is said to be solution of system of linear equation system of linear equation all right

so a system of linear equation can have unique solution it can have infinitely many solutions and it can also have no solution i have no solution okay

so let's see some examples consider system of equations two equations into unknown given by x one plus two x two is equals to four to x one plus x two is equals to 2

so this is the system of equations

so it is easy to see that x_1 is equals to 0 and x_1 is equals to x_2 is equals to 2 is unique solution is the uni in unique solution genuine solution of one all right let's see another example for example consider this x 1 plus x 2 is equals to 2 2×1 plus 2×2 is equals to 4 okay if you consider this system of equations

so it is easy to see that the second equation can be obtained by multiplying first equation by two all right

so in this system of equation any solution α and 2 minus α where α belongs to some real number by real number is a solution a solution of two that is two as infinitely many solution ok

so let us consider another example at x 1 plus x 2 is equals to 2 2×1 plus 2×2 is equals to 3

so this system of equation has no solution it's very clear because let me just delete this okay

so this system has no solution because if x_1 x plus x_2 is 2 then $2x_1$ plus $2x_2$ will be 4 not 3 .

so this system has no solution okay
so okay

so we have seen the examples where the system has unique solution where a system has infinite solution where the system has no solution now the question is the question is how do we decide how do we decide how do we decide about the the how do we decide about the solution of a system of linear equation $ax = b$ where a is $n \times n$ matrix x is $n \times 1$ vector b also $n \times 1$ vector okay

so how do we decide it

so there are some condition

so which are given in terms of the rank of a matrix

so that gives that decision the system has whether system has a unique solution or the system has infinite many solution or system has no solution okay

so let's see what are those conditions first one is okay

so if rank of a is equals to rank of augmented matrix $a|b$ is equals to n then system of equations has unique solution in this case determinant of a is not equal to zero second condition is if rank of a is equals to rank of a augmented matrix augmented matrix $a|b$ is equals to m less than n then system has infinite demanding solution all right the condition for the no solution is if rank of a is not equal to rank of augmented matrix maybe then the system of equations has no solution okay

so let me recall the definition of rank

so rank of a can be obtained by reducing matrix a using elementary row operation to each loan form where number of non-zero rows of matrix in each long form gives the rank of a matrix okay

so let's give some examples of this rank i mean each learn for matrix example is one two three zero two five zero zero minus one

so this matrix is in island form because if you see the zeros are in the increasing order if the second row has one zero then the third row has 2 0 okay

so the rank here is rank of this matrix is 3 because you have all rows are nonzero second example is let's say this is the matrix 1 3 0 0 0 2 0 0 zero zero see here also in in the second row there are two zeros

so the third row has three zeros for this rank is two because the last row is a zero row and the number of non-zero rows is two because the number of non-zero rules two okay

so so this is how i mean

so we use the elementary row operation to reduce the matrix into this long form to calculate the rank of a matrix okay

so i think yeah

so we are we are done with the necessary background on this system of linear equation

so now we'll solve some examples based on these concepts

so let's solve some problems related with system of linear equation question let α, λ, μ belongs to \mathbb{R} that is set of real numbers consider the system of linear equations $\alpha x + 2y = \lambda$ okay for what values of α, λ, μ in the system of linear equations questions have a unique solution unique solution infinitely many solutions and last part is unknown solution

so we need to derive a condition on the α, λ, μ for which we can say that when it will have a unique solution when we it will have infinitely many solutions and when it has no solution okay

so let's solve this problem okay

so for this system of linear equation we have this matrix a which is given by $\alpha \ 2 \ 3$ minus $2 \ b$ is $\lambda \ \mu$ okay

so let's solve the first part

so for unique solution the condition is that determinant of a is not equal to 0 and which is same as the rank of a augmented b is equals to rank of a is equals to 2

so both these transitions are equivalent all right

so let's find the determinant of a which is nothing but minus 2 alpha minus 6 and this is not equals to 0 this implies alpha is not equals to 3 minus 3 alpha is not equal to -3

so here we don't have any condition on lambda and mu

so we can say that for this implies for alpha is equals not equals to minus 3 and lambda mu belongs to R any any real number system will have unique solution ok

so let's solve the second part

so a matrix is $\begin{bmatrix} \alpha & 2 & 3 \\ \lambda & \mu & \mu \end{bmatrix}$ b is $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$

so we need to derive a condition on alpha lambda mu for which the system has a system has infinitely many solution for that let's consider the augmented matrix okay

so this is nothing but alpha 2 we append this vector v b also here and three minus two all right

so here if i if i apply the row operation r2 is replaced with the r2 plus r1 then we get alpha 2 and here it is lambda and then you get 3 plus alpha 0 and i think i need to erase this okay here it will be lambda plus mu

so this is the this is the reduced form of augmented matrix

so ok

so now for infinitely many solutions for infinitely many solution rank of augmented matrix a b should be equals to the rank of a and that should be less than 2 all right

so all right

so here if alpha plus 3 is equals to 0 and lambda plus mu is equals to 0 that is alpha is equals to minus 3 and lambda is equals to minus mu okay then rank of a augmented b is equals to rank of a is equals to 1 okay because the last row last row is 0 all right and this implies that the system of equations will have infinite limit solutions

so this is the condition for these values for minus 3 and lambda is equal to minus mu you will have infinitely many solution

so let's go back to the third part

so c part is that we need to derive a condition for which there is no solution right

so as we have seen that a augmented matrix b reduce equivalent to alpha 2 lambda 3 plus alpha 0 lambda plus mu okay

so here if alpha is equals to minus 3 okay let me just yes okay alpha is minus 3 and lambda is not equals to minus mu that means lambda plus mu is not equal to zero this implies that rank of a augmented b this matrix okay is equals to 2 and rank of a is equals to 1 because a matrix last row will be will be 0 but if you see the last row of augmented matrix a b you'll have entry 0 0 and there is a non zero into lambda plus mu because lambda plus mu is not equal to zero

so this is the condition this is the condition of no solution that is rank of a not equals to rank of a augmented b this implies that the system has no solution okay

so we have derived the condition for all the all the cases it's all another problem for a real number alpha if the system one alpha alpha square alpha one alpha alpha square alpha 1 x y z is equals to 1 minus 1 one of linear equation the system of linear linear equations has infinitely many solutions then what will be done the value of 1 plus alpha plus alpha square

so this is the question

so let's solve it answer okay

so first consider the augmented matrix $a \ b$ this is nothing but one α α^2 α^3 α^4 1 1 1 1 okay

so let's apply some row transformation

so this is the equivalent to if i if i apply let's say i take the r_2 and then play $r_2 - \alpha r_1$ and r_3 replace with $r_3 - \alpha^2 r_1$ okay then there is no change in the first row y square this is 1 and then this is 0 and this is $1 - \alpha^2$ and this is $\alpha - \alpha^3$ all right then this is 0 this is $\alpha - \alpha^3$ okay and this is $1 - \alpha^4$ to the power 4 and this one will be $1 - \alpha$ and this is $1 - \alpha^2$ okay

so to decide on the rank we just we need to make one more zero in the third row okay

so let's let's see what what operation we should do

so here this is equivalent to okay

so i will apply that r_3 i will replace with the $r_3 - \alpha r_2$ then okay first there is no change second also there is no change and the third this is zero and this will also be 0 because yeah and then this is nothing but $1 - \alpha^4$ this is $1 - \alpha^4 - \alpha^2(1 - \alpha^2) + \alpha^2$ to the power 4

so $1 - \alpha^2$ and this is $1 - \alpha^2 + \alpha^2 - \alpha^4 + \alpha^2$ this is α^2

so this is one plus α okay

so now this is the this is the reduced reduced form of augmented matrix okay

so okay

so let me

so this is what we we have the augmented b is nothing but one and one α α^2 α^3 α^4 1 0 $1 - \alpha^2$ $\alpha - \alpha^3$ $1 - \alpha^4$ $1 - \alpha$ $1 - \alpha^2$ $1 - \alpha^4 - \alpha^2(1 - \alpha^2) + \alpha^2$ and this is $1 + \alpha$

so since the system has infinitely many solutions

so this means that rank of a augmented b is equals to rank of a and which is less than 3

so now when is it possible

so the rank of augmented matrix will be 2 if $1 - \alpha^2 = 0$ and $1 + \alpha \neq 0$

so if $1 - \alpha^2 = 0$ and $1 + \alpha \neq 0$ okay

so if these two conditions holds together then the third row of the augmented matrix is 0 and the third row of a matrix is automatically 0 and then the rank of a is equals to rank of a augmented b will be 2 .

so in under this

so together these conditions implies that α value is -1 okay and at this value rank of a augmented b is equals to rank of a is equals to 2 less than 3 okay

so for α is equals to -1 this implies system under this condition this is given

so this implies that for α is equals to -1 system has infinite solutions okay and this implies that $1 + \alpha + \alpha^2$ will be 1

so this is the this is the final answer

so let's solve another problem okay

so consider the system of equation $x - 2y + 3z = -1$
 $-x + y - 2z = k$
 $x - 3y + 4z = 1$ then for what values of k system of equations has no solution okay

so let's solve this problem okay

so consider the augmented matrix $a \ b$
 so where is $1 \ -2 \ 3 \ -1 \ -1 \ 1 \ -2 \ k \ 1 \ -3 \ 4 \ 1$ okay
 so let's reduce this system
 so let's apply the following transformation we do $r_2 \ r_2 + r_1$ and $r_3 \ r_3 - r_1$
 so what do we get we get
 so this is the our radio system the first row there is no change one minus 2 3
 minus 1 and then 0 minus 1 then 1 minus 3 is minus 2 and $k - 1$ we applied $r_2 + r_1$ sorry okay
 so it is 0 i mean we add minus 1 plus 1 1 1 minus 2 minus 1 minus 2 plus 3 it's
 a 1 and the $k + 1 - 1$ now next one was $r_3 - r_1$
 so $r_3 - r_1$ this entry is 0 minus 3 plus 2 minus 1 and 4 minus 3
 so 1 and 1 plus 1 2 all right ok
 so let's reduce it further okay
 so now i will here what i will do is i will just um okay here i will apply the
 transformation which is nothing but $r_3 + r_2$ then after this
 transformation we get 1 minus 2 3 minus 1 there is no change in the second row 1
 $k - 1$ then this becomes 0 um sorry i think no not plus it should be $r_3 - r_2$
 minus r_2 okay then this becomes 0 and this is this becomes 0 all right and then
 this will becomes $3 - k$ ok
 so this is what
 so now now what is the rank of a here rank of a is anyway two all right
 so if you want system to have no solution then for system to have no solution
 we need rank of augmented matrix to be 3 this implies $k \neq 3$
 so because if $k \neq 3$ then the last row of augmented matrix is not
 0 and the rank of a argument in b this matrix is 3 all right
 so so
 so this implies that for $k \neq 3$ system will have no solution
 so this is the final answer okay students i will stop here now thank you for
 attending this session in the next session i will solve some more interesting
 problems based on system of linear equations thank you you