

hello students welcome to iit palm mathematics problem solving session this is lecture number two and our topic is matrices and determinant in today's lecture i will solve some more problems related with matrices and determinant the required background for these problems is already being given in the first lecture okay

so let's start with question number one question okay if the joint of and joint of a 3 cross 3 matrix b is 1 4 4 2 1 7 1 1 3 then what are the possible values of the determinant of p okay

so this is the question

so let's solve this problem

so answer okay

so as we know that for an n cross n matrix a determinant of adjoint a is equals to the determinant a to the power n minus 1.

okay and we have a joint p is given as 1 4 4 2 1 7 1 1 3

so this is the adjoint of p all right okay

so this is

so then what will be the determinant of this i joined the p

so determinant of a joint of p is given by 1 into 3 minus 7 okay n minus 4 into 2 into 3 6 minus 7 plus 4 times 2 minus 1.

okay

so this is nothing but

so this is minus 4 this is plus 4 and then this is plus 4 and this is equal to 4

so given n is equals to 3 determinant of adjoint of p is equals to determinant of a of whole square this implies you determine sorry and this is the p okay this is determinant p this implies determinant p whole square is equals to 4 implies determinant of p is equals to plus minus 2

so let me just check in the first slide okay this is this i have written it for the any general a matrix this formula all right okay

so this is the this is the final answer

so determinant of p has two values plus and minus two let's solve another question ok

so for 3 cross 3 matrices m and n show that

so that first part is n transpose m n is symmetric or skew symmetric according am is symmetric or excuse me okay let's write this in another line or symmetry

so this is the part one

so what it says that and transpose we need to show that n transfer german is symmetric if m is symmetric it is still symmetric if m is exclusive second part is m into n minus n into m is skew symmetric for all symmetric matrices m and n okay

so how to solve this problem okay

so let's solve it answer

so let's solve the first part okay

so first suppose m is a symmetric matrix this implies m transpose is equals to m now we need to show that n transpose m n is symmetric matrix

so let's take its transpose that means we need to show that the transpose of this matrix is itself

so okay

so let's make the pair uh for example the first one is the n transpose we take it as a one matrix m into n we take it as another matrix and it's transpose and this is nothing but m and transpose and transpose of transpose because we know that a b transpose is nothing but b transpose into a transpose

so now again we can apply the transpose formula in this project of matrix

so this is nothing but n transpose m transpose and n and because m transpose is

m

so this is nothing but  $n^T m n$

so what did we show we showed that  $n^T m n^T$  is nothing but  $n^T m n$

so this implies that  $n^T m n$  is a symmetric matrix okay

so now if we suppose that image is two symmetric okay

so another part of the part one itself let  $u$  be skew symmetric this implies  $m^T$  is equal to  $-m$  okay then again we do the same thing we take the  $n^T m n$  and we take its transpose and we know that from in the previous slide we this is nothing but  $n^T m^T n$  and now in place of  $m^T$  we write it as  $-m$

so this is nothing but  $-n^T m n$

so this implies  $n^T m n$  is a skew symmetric matrix sorry matrix ok

so so this is how we prove the first part now let us prove the second part here we need to show that if  $m$  and  $n$  are symmetric matrices then we need to show that  $m n - n m$  is skew symmetric okay

so let  $m$  and  $n$  be symmetric matrices this implies  $m^T = m$  and  $n^T = n$  okay

so now let's take the this matrix  $m n - n m$  and then take the transpose as we know that  $(a - b)^T = a^T - b^T$  is nothing but  $a^T - b^T$

so this is nothing but  $m^T n^T - n^T m^T$  this is nothing but  $n^T m^T - m^T n^T$  here the product formula we use and this is nothing but  $m^T n^T - n^T m^T$  and this is nothing but  $n^T m^T - m^T n^T$

so  $n^T m^T$  is  $m n$  this is  $m n - m n$

so this is nothing but  $-m n$  we take a common  $m n - n m$

so this implies what

so  $m n - n m$  transpose is the negative of the  $m n - n m$

so this implies this implies  $m n - n m$  is a skew symmetric matrix ok

so okay

so this is how we prove the both the parts okay

so let's solve another problem question let  $x$  and  $y$  be two arbitrary three cross three non-zero skew symmetric matrices and  $z$  be an arbitrary three cross three non-zero symmetric matrix then show that  $y^3 z^4 - z^4 y^3$  is a symmetric matrix part  $x^4 + y^4$  is a symmetric matrix part  $x^4 z^3 - z^3 x^4$  is a skew symmetric by  $d x^{23} + y^{23}$  is a skew symmetric matrix let me just erase this okay

so so this this is the problem

so let's let's solve it okay

so okay

so it is given that  $x^T = -x$  and  $y^T = -y$  and  $z^T = z$  because  $x$  and  $y$  are skew symmetric matrices and  $z$  is a symmetric matrix okay

so let's see what can we say about  $x^n$  transpose what  $y^n$  transpose the  $g^n$  transpose for any natural number  $n$  okay

so let's see  $x^n$  transpose is nothing but  $x$  multiplied by  $n$  times transpose and this is  $-n x$  okay

so let's use that formula the product of a transpose which transpose into  $x^T x^T$  and this is nothing but  $-x$  okay

so let's say that this is nothing but  $x^n$  and this is nothing but  $-x^n$

so this is nothing but minus 1 to the power n x to the power n  
 so this is the relation we have that means x to the power n transpose is equals  
 to minus 1 to the power n x to the power n  
 so this means x transpose this means x to the power n transpose will be equals  
 to x to the power n if n is even and this is minus x to the power n if n is odd  
 so the same thing is true for y because y is also symmetric  
 so y to the power n transpose is equals to y to the power n if n is even this  
 is minus y to the power n if n is odd  
 so now let's see what we can say about z to the power n transpose  
 so this is nothing but z z multiplied by z and times okay okay  
 so this is nothing but z transpose the z transpose multiplied by z transpose  
 and this is nothing but z transpose to the power and energy transpose is z  
 so this is nothing but z to the power n  
 so we have z to the power n transpose is equals to z to the power n for all and  
 okay  
 so let's use these relation in proving these all these four parts  
 so let's go to the party ok  
 so it says that y this is the matrix we need to investigate y cube z 4 minus z  
 4 y cube okay  
 so let's take transpose this is nothing but y cube z to the power 4 transpose  
 minus z to the power 4 y cube transpose okay  
 so now again the product 1 this is nothing but z to the power 4 transpose y  
 cube to the power y cube transpose minus y cube transpose z 4 transpose because  
 z was the z was the symmetric matrix  
 so any power of z and its transpose will give the the same z to the power 4  
 okay here 3 is odd number  
 so y cube transpose will be nothing but minus y cube this is minus minus y cube  
 and this is z 4  
 so this is nothing but  
 so this will be plus nothing but y cube z 4 minus z 4 y cube all right  
 so this means this is same as this right  
 so that means this matrix is symmetric matrix this implies y cube z 4 minus z 4  
 y cube is a symmetric matrix is a symmetric matrix okay  
 so let's go to the another part which is a part b  
 so here we need to investigate this x to the power 44 plus y to the power  
 so let's take a transpose  
 so this is nothing but equals to x to the power 44 transpose plus y to the  
 power 44 transpose okay  
 so because 44 is a even number  
 so that means this is nothing but x to the power 44 and this is y to the power  
 44 this implies x to the power 44 plus y to the power 44 is a symmetric matrix  
 ok  
 so now go to the c part which is also similar x to the power 4 z cube minus z  
 cube x to the power 4  
 so let's check this take transpose and this is nothing but x to the power 4 c  
 cube transpose minus z cube x to the power 4 transpose this is equals to z cube  
 transpose x 4 transpose minus x 4 transpose z q all right  
 so this is nothing but z cube this is nothing but x 4 minus x 4 to the power 4  
 z cube and this is nothing but the minus of x to the power 4 z cube minus z cube  
 over 2  
 so this means this means this matrix this implies that x to the power 4 z cube  
 minus z cube x to the power 4 is a skew symmetric matrix okay  
 so let's go to the last part part d which is also simple it is x to the power  
 23 plus y to the power 23

so if you take a transpose because  $x$  and  $y$  both has the odd power  
so this is nothing but  $x$  to the power 23 transpose plus  $y$  to the power 23  
transpose and this is nothing but minus  $x$  to the power 23 minus  $y$  to the power 23

so which is again minus times  $x$  to the power 23 plus  $y$  to the power 23  
so this implies that  $x$  to the power 23 plus  $y$  to the power 23 is skew symmetric matrix okay

so this is it  
so these type of problems you can you can always you can always solve just by looking at this power of  $x$  and  $y$  whether it is odd power or even problem we can make a use of that to solve these type of problems okay

so let's solve another problem let  $k$  be a positive real number okay and let  $a$  be a matrix  $3 \times 3$  matrix defined as  $k$  minus  $1$   $2\sqrt{k}$   $2\sqrt{k}$  second row is  $2\sqrt{k}$   $1$  minus  $2k$  minus  $2\sqrt{k}$   $2k$  minus  $1$  okay and there is another matrix  $b$  which is defined as  $0$   $2k$  minus  $1$   $\sqrt{k}$   $1$  minus  $2k$   $0$   $2\sqrt{k}$  okay third row is minus  $\sqrt{k}$  minus  $2\sqrt{k}$   $0$  okay if determinant of  $a$  plus determinant of adjoint of  $b$  is equals to  $10$  to the power 6 then what is the value of square bracket  $k$  where square bracket  $k$  denotes the largest integer less than or equal to  $k$  ok ok

so the matrices  $a$  and  $b$  are given and there is a relation given and we need to find the value of square root of  $k$

so let's solve this problem okay

so first let's calculate the determinant of  $a$  okay

so what will be the determinant of  $a$  okay

so let me just write it down  $2k$  minus  $1$   $2\sqrt{k}$   $2\sqrt{k}$   $2k$  minus  $1$   $2\sqrt{k}$   $2k$  minus  $1$

so let's expand this determinant from first row okay this is equals to  $2k$  minus  $1$  multiplied by  $1$  minus  $1$  sum minus  $1$  plus  $4k$  square okay minus  $2\sqrt{k}$  then we find the determinant of the sub matrix which is minus  $2\sqrt{k}$  and minus  $4k$  root  $k$  okay plus  $2$  times root  $k$  into  $4$  times  $k$  root  $k$  plus  $2$  times okay okay

so let's open this okay

so  $2k$  minus  $1$  multiply minus  $1$  it's  $1$  minus  $2k$  plus  $8k$  cube minus  $4k$  square plus  $4k$  plus  $8k$  square okay plus  $8k$  square square plus  $4k$  okay

so this is equals to  $8k$  cube okay then in  $k$  square is  $16$  minus  $4$

so  $12$  plus  $12k$  square okay and then  $k$  what is the four point eight minus two plus six  $k$  plus one ok

so this is the value of determinant  $a$  okay since  $k$  is greater than equals to zero this implies in fact other than it's a positive

so okay yeah  $k$  is strictly greater than  $0$  this implies determinant of  $a$  is strictly greater than  $0$  all right ok

so now let's calculate the determinant of  $b$  okay

so this is nothing but the if we expand from the first row all right okay

so let me let me just write the determinant for you  $0$  to  $k$  minus  $1$  root  $k$   $1$  minus  $2k$   $0$   $2\sqrt{k}$  minus root  $k$  minus  $2\sqrt{k}$   $0$  all right

so yeah

so if you expand from the first row then first entry is  $0$  then minus  $2k$  minus  $1$  multiplied by  $2$  times  $k$  all right plus root  $k$  and then you multiply with the minus  $2\sqrt{k}$  plus  $4k$  root  $k$  all right

so let's expand this

so this is nothing but minus  $4k$  square plus  $2k$  minus  $2k$  plus  $4k$  square all right

so this is nothing but  $0$

so the determinant of  $b$   $0$  all right

so yeah

so we have a relation which is given what is that this is nothing but determine what is that determinant of a joint of a plus determinant of adjoint of b is equals to  $10$  to the power  $6$

so we have we know the relation between determinant of a joint a in terms of determinant a given is a  $3 \times 3$  matrix

so determinant of a joint a is nothing but determinant a square

so this is nothing but determinant of a square plus determinant of b square this is equals to  $10$  to the power  $6$

so determinant b is  $0$

so this is nothing but determinant of a square is equals to  $10$  to the power  $6$  given determinant is greater than  $0$  this implies determinant of a is equals to  $10$  to the power  $3$  okay

so now let's write the expression of determinative which we calculated which was nothing but  $8k^3 + 12k^2 + 6k + 1$  this is equals to  $1000$

so this implies  $8k^3 + 12k^2 + 6k - 999$  is equal to zero

so basically we need to find the root of this okay

so which may be little hard to find

so let's uh denote this expression on the left hand side by  $b_k$   $b_k$  is equals to  $8k^3 + 12k^2 + 6k - 999$

so let's check this value for different value of  $k$  okay

so let's okay

so yeah sorry for that yeah

so  $p_1$  what is the  $p_1$  value

so  $p_1$  we know

so  $p_1$  is nothing but  $8 + 12 + 6 - 999$  this is definitely less than  $0$

so what is  $p_2$

so  $p_2$  is nothing but  $8k^3$

so  $64 + 12$  let me just see what is  $p_2$  you know  $12 \times 4 + 6 \times 2 - 999$

so this is also definitely less than  $0$  what about  $p_3$   $p_3$  is  $8 \times 27 + 12 \times 9 + 6 \times 3 - 999$  and this is also turned out to be less than  $999$  okay

so what about the  $p_4$   $4^4$  is  $8 \times 64 + 12 \times 16 + 6 \times 4 - 999$

so let's calculate what is this  $5^4 - 999$   $5^4$  is  $625 + 12 \times 25 + 6 \times 5 - 999$  minus  $999$  sorry this roughly this is  $700$  something which is definitely less than  $999$

so this is also less than zero okay

so let's see what is  $p_5$   $p_5$  is  $8 \times 125 + 12 \times 25 + 6 \times 5 - 999$

so this is nothing but  $1000 + 300 + 30 - 999$  this is greater than  $0$

so so basically between  $4$  and  $5$  after  $4$  it changes the sign this  $p_k$  that polynomial changes the sign

so this implies that there will be a  $p_k$  will be  $0$  for sum  $k$  lies between  $5$  and  $4$  okay

so this means what because this implies  $k$  which is the largest integer less than or equals to  $k$

so which will be nothing but  $4$  okay because there is a  $0$  which lies between  $4$  and  $5$  okay

so this is the this is the final answer okay okay

so let's solve another problem question let  $p$  be a  $3 \times 3$  matrix given by  $3 \times 3$  matrix  $\begin{pmatrix} 1 & \alpha & 0 \\ \alpha & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

so this is a  $3 \times 3$  matrix where  $\alpha$  belongs to some real number suppose  $q$  be another matrix matrix such that  $p^{-1}q$  is equals to  $k$  times  $I$  where  $k$

belongs to  $r \times k$  not equals to  $\emptyset$  and  $i$  is three cross three identity matrix is the identity matrix okay

so if  $q$  of  $2 \times 3$  is equals to  $\text{minus } k \text{ by } 8$  and determinant of  $q$  is equals to  $k$  square by  $2$  then

so that point is determinant of  $p$  at joint  $q$  is equals to  $2$  to the power  $9$  and point  $b$  is  $4\alpha - k + 8$  is equals to  $\emptyset$

so given all this information we need to we need to prove this okay

so certainly we need to find this value of  $\alpha$  and  $k$  and

so let's see let's see how do we solve this problem okay answer

so it is given that  $p$  into  $q$  is nothing but  $k$  times identity matrix

so if you take determinant on both sides we get determinant of  $k$  times  $i$  and this since  $i$  is a  $3 \times 3$  matrix

so this is nothing but  $k$  cube and determinant of  $i$  which is equals to  $1$  and this is equal to  $k$  cube and because  $k$  is not  $\emptyset$  this implies this is not equal to  $\emptyset$  because  $k$  is not equal to  $\emptyset$

so we have determinant of  $p$  into we can apply this product formula of determinant determinant  $q$  not equals to zero

so from here we observe that determinant of  $p$  is not equal to zero and determinant of  $q$  is also not equal to  $\emptyset$  okay

so let's see what is a  $p$  matrix

so  $p$  matrix which is given is  $3 \text{ minus } 1 \text{ minus } 2 \text{ } 2 \text{ } 0 \text{ } \alpha \text{ } 3 \text{ minus } 5 \text{ } 0$

so if you calculate the determinant of  $b$  this is nothing but okay

so let's uh let's take the column column third column because there is one zero so let's expand from there

so this is nothing but  $\text{minus } 2$  times okay this is  $\text{minus } 10$  and this is this will be  $3 \text{ plus } 2 \text{ } 5$

so this is  $\text{minus } \alpha$  times  $\text{minus } 15$  plus  $3$

so this is nothing but we get  $12\alpha + 20$  okay given determinant  $p$  is not  $\emptyset$  this implies that  $12\alpha + 20$  is not equal to zero because determinant of  $p$  is not equals to zero okay

so now again again we go back to go back to this relation  $p$  into  $q$  is equals to  $k$  times  $i$  this implies that as we did in the beginning determinant  $q$  is nothing but  $k$  cube all right

so this implies that the determinant of  $q$  is nothing but  $k$  cube divided by determinant  $p$  which is nothing but  $12\alpha + 20$  all right okay

so determinant  $q$  we know now okay

so now again from this relation  $p \times q$  is equals to  $k$  times  $k$  times  $i$  we have you can multiply  $p$  inverse on both sides

so because determine  $p$  is not zero

so  $p$  inverse exist this implies that  $q$  is nothing but  $k$  times  $p$  inverse

so to calculate the  $p$  inverse we should know what is the joint of  $p$

so a joint of  $p$

so this is nothing but  $k$  times a joint of  $p$  divided by determinant  $p$

so let's calculate a joint of  $p$

so joint of  $p$  is easy to calculate it is nothing but the transpose of co-factor matrix

so let's calculate it okay

so so first entry will be  $5\alpha$  and then  $3\alpha$  this is  $\text{minus } 10$  this is  $10$  this is  $6$  this is  $12$  this is  $\text{minus } \alpha$  this is  $\text{minus } 3\alpha$  plus  $4$  and this is  $2$  all right

so this is the this is easy to calculate

so now now let's see what is  $q$   $q$  is nothing but  $k$  times  $d$  divided by determinant  $p$  which is nothing but  $12\alpha + 20$  and  $i$  joined  $p$  which is nothing but the same matrix  $5\alpha \text{ } 10 \text{ minus } \alpha \text{ } 3 \text{ } \alpha \text{ } 6 \text{ minus } 3 \text{ } \alpha \text{ plus}$

4 this is minus 10 12 and 2 okay

so this is the q matrix and given we know that  $q_{23}$  is minus k by 8

so let's see what is  $q_{23} q_{23}$  will be the 2 3 entry of this matrix

so ok let's see given  $q_{23}$  is nothing but minus k by 8

so this implies this is nothing but minus 3 alpha plus 4 times k divided by 12 alpha plus 20 this is nothing but minus k by 8

so because given k is non-zero we can we can cancel on both sides we have minus this k and k will be cancelled out minus minus will be cancelled out

so we will get 3 alpha plus 4 divided by 12 alpha plus 20 is equals to 1 by 8 this is nothing but 24 alpha plus 32 is equals to 12 alpha plus 20

so this is nothing but 12 alpha is equals to minus 12.

all right this implies that alpha is equals to minus 1

so we obtain the value of alpha which is nothing but minus 1 all right okay

so let's use some more relations to find out the value of k okay

so another relation we have is determinant of q is given as a k square by 2 all right

so what is the determinant of q we know that it is we calculated in the beginning k cube divided by 12 alpha plus 20 this is equals to k square by 2 again k is not 0

so we can cancel on k square from both sides

so k square is gone near 3 will be gone

so this is nothing but this is nothing but 2 times k is equals to 12 alpha is minus 1 plus 20

so this implies that 2 k minus 2 this is 8 21 is 12 this implies that k sorry is 4 implies that k is equals to 4 okay

so we know that we know this that k is 4 alpha is minus 1 then the part b we can we can just verify that 4 alpha minus k plus 8 this is nothing but 4 into minus 1 this is minus 4 plus 8 this is nothing but 0 all right

so the part b is part b is done okay

so let's let's go to part one okay

so the part is that we need to find the value of determinant of p at joint of q all right

so this is nothing but we first use the determinant product of the determinant is nothing but for determinant of product of two matrices nothing but product of the determinant determinant p and two determinant of a joint of q and determinant of p is nothing but 12 alpha plus 20 and this is nothing but uh determinant by join q is nothing but determinant of q whole square because q is also 3 cross 3 matrix okay

so what is this 20 and determinant q was k cube divided by 12 alpha plus 20 whole square

so this is nothing but k to the power 6 divided by 12 alpha plus 20 and alpha is minus 1

so here k is 4

so 4 to the power 6 and this is 12 plus 20 is nothing but 8

so this is nothing but 2 to the power 12 here 2 to the power 3 and this is nothing but 2 to the power 9

so we get we get determinant of p into a joint of q is nothing but 2 to the power 9

so this is what we had to we had to prove okay

so yeah

so that's it

so so this was the this was the solution of this question we verified both the parts okay students

so i'll stop here now thank you for attending this session i will solve some

more interesting problems on matrices and determinant in the next lecture thank you you

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