

hello students politics problem solving session this is lecture number one and our topic is matrices and determinant i will expect you to have a basic background on matrices and determinants to follow these lectures

so although although i'll topic before starting problem solving session and properties of matrices and determinants let a be m cross n matrix defined by a_{ij} equals to $a_{11} a_{12} \dots a_{1n} a_{21} a_{22} \dots a_{2n} \dots a_{m1} a_{m2} \dots a_{mn}$ this is how we define a matrix okay in short we write a as a square bracket a_{ij} okay then let's define the transpose of a matrix a is denoted by a^T and it is defined by interchanging the row and column that means a^T equals to $a_{11} a_{21} \dots a_{m1} a_{12} a_{22} \dots a_{m2} \dots a_{1n} a_{2n} \dots a_{mn}$ similarly the second column is replaced by the second row $a_{21} a_{22} \dots a_{2n}$ similarly the m th column is replaced by the m th row $a_{m1} a_{m2} \dots a_{mn}$.

okay

so when m equals to n and a is square matrix okay then a is said to be symmetric if a^T equals to a okay

so that is a_{ij} equals to a_{ji} right for all i, j okay similarly then again when we define we say a is antisymmetric if a^T equals to minus a this implies a_{ij} equals to minus a_{ji} for all i and j this implies a_{ii} equals to minus a_{ii} implies a_{ii} equals to 0 for all i okay

so guys 4×4 matrix

so for example if c is a scalar and then c times a is defined as ca into e_{ij} Taylor by a matrix then each element suppose or not necessary the square transpose uh square one but you take any two matrices a and b $(a+b)^T$ is same as $a^T + b^T$ and product of two matrices and its transpose is same as $b^T a^T$ okay

so let's define another notion which is called the determinant of matrix okay editor okay

so determinant of matrix is a scalar quantity and it is defined for a square matrix

so let a be n cross n matrix then determinant of a is a scalar quantity okay and then we also denote determinant of a by $|a|$ you just put two parallel lines and put matrix inside that okay

so let's define this okay

so suppose when n equals to 2

so the a is represented by $a_{11} a_{12} a_{21} a_{22}$ then determinant of a okay is defined by $a_{11} a_{22} - a_{12} a_{21}$ okay when n equals to 3 then a is $a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33}$ okay then how do we define the determinant

so determinant of a

so i will use the both the notation either the $|a|$ with two parallel lines or just a determinant Δ

so this is defined by we just take the first row okay and take the first element a_{11} then we get the sub matrix by deleting the first row and first column and then we calculate the determinant

so $a_{22} a_{33} - a_{23} a_{32}$ then minus a_{12} the second one and then we delete the first row and the second column and whatever the sub matrix we get we just write it here $a_{21} a_{33} - a_{31} a_{23}$ and a_{13} third one and then we delete the first row and the third column and we get the following sub matrix $a_{21} a_{32} - a_{31} a_{22}$ okay

so so this is how we calculate a determinant of 3×3 matrix similarly we can find the determinant of any $n \times n$ matrix

so here i took the first row to expand it we can use any row and any column but only thing is this multiply this the sign of these multipliers like $a_{11} a_{12} a_{13}$ it's defined by the following

so sign of a_{ij} is given by the sign of $(-1)^{i+j}$

so if $i + j$ is odd number then there will be negative sign otherwise it will be positive sign okay

so for example we can compute the determinant by any by taking any row and any column for example if a is let me just write once a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} okay

so suppose instead of a first row i take the third row okay third row if i take then i can write the determinant of a is equals to the the first element of the third row is a_{31} and the sign will be positive because three plus one minus one to the power four is one

so plus sign

so we take the first element a_{31} and delete the third row and the first column and we get some matrix a_{12} a_{22} a_{13} a_{23} then we take the second one a_{32} and this sign will be negative because three plus two is odd number five and then we delete the third row and the second column we get a_{11} a_{13} a_{21} a_{23} and then the third entry a_{33} plus 3 is an even number

so i'll delete the third row and third column and we get a_{11} a_{12} a_{21} a_{22}

so in a similar way in a similar way we can calculate the determinant of any matrix by considering any row and any column okay all right

so let's see some properties of determinant

so i'll list out some important properties of determinant which will be useful while solving the problems related with determinant

so first one is the determinant of a is equals to determinant of a transpose second is if any two rows or columns of a determinant are interchanged then sign of determinant changes third one if all the elements of a row or column of a are zero then determinant of a is zero fourth property if any two rows or columns of a matrix a are identical and then determinant of a is zero

so some more properties fifth one suppose if suppose in a determinant in the first row you multiply with the same constant k of a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} suppose we encounter the determinant

so this determinant can be written as k times the determinant of a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} okay

so the sixth property if suppose we have the following determinant a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} plus x a_{11} a_{12} a_{13} a_{21} a_{22} plus y a_{11} a_{12} a_{13} a_{21} a_{22} plus z and a_{31} a_{32} a_{33} this determinant can be written as a sum of two determinants first one is a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} this is the first one then plus the second one is a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33}

so this is how we break this first determinant as a sum of 2 determinant okay

so so seventh property okay

so it is a product of two matrix determinant of a product of two matrix suppose a and b both are square matrix okay of order n then determinant of ab as a determinant of a into determinant of b okay

so these are the these are the some properties one more one more important property let me just list it for example what is the determinant of c times a matrix a where c is scalar

so this is nothing but c^n determinant of a given is a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33} and matrix okay

so yeah

so these are the some important properties of uh determinant if some more properties are used then i'll explain i'll explain when i when i solve those problems okay

so let me introduce one more one more concept in my research it's called inverse of a matrix okay

so let a and b and then cross in matrix then the inverse is defined by a^{-1} is

equals to a joint of a divided by determinant of a where determinant of a is non-zero okay

so this means a inverse exists only if determinant of a is not equals to zero okay

so we call invertible matrix as non-singular matrix we call an invertible matrix as non-singular this is another name non-singular matrix okay

so so we know how to calculate the determinant

so now what is the adjoint

so adjoint of a and this is nothing but the transpose of co-factor matrix now the question is what is cofactor okay

so let me explain all these things with the three cross three matrix suppose a is a three cross three matrix a one one a one two a one three a two 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3 ok

so what is the what is the adjoint adjoint of a

so if i write a joint of a i just write it as a 1 1 a 1 2 a 1 3 a 2 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3

so where this a i j is this is the i j co factor okay and

so this matrix you know

so a joint is nothing but the this is the cofactor matrix and you just take its transpose okay now the question is how do we how do you compute these cofactors okay

so in general this a ij is nothing but minus 1 to the power i plus j into determinant of sub matrix obtained by deleting it row and jth column for example e 1 1 is the 1 1 co factor which is obtained by deleting first one first column and this is the determinant this is defined as a determinant of a sub matrix obtained by deleting first one first column

so what do we get we get a 2 2 a 2 3 a 3 a 3 3

so this is how a 1 1 is defined similarly we can we can define let's say the other other aijs

so let's see what is a 1 2

so a 1 2 is nothing but you delete the first row and first row and the second column

so you get a 2 1 a 2 3 a 3 1 a 3 3 and here it will be multiplied with the minus 1 to the power 3

so this is the minus sign

so similarly similarly other other aij can be calculated

so to find the joint a we just first calculate all these cofactors and then we form a co-factor matrix and take its transpose that will be the adjoint of a okay

so now we know how to calculate the inverse inverse of a matrix okay

so there is one important okay

so there are two properties related with the inverse

so for example first one is the determinant of a inverse is equals to 1 upon determinant of a

so how do we get it a inverse is equals to identity matrix then the determinant of a inverse is equals to determinant of i which is g equals to 1 and a i identity matrix all right then this is nothing but determinant a into determinant a inverse equals to 1 implies determinant of a inverse is equal to 1 upon determinant of a okay

so now another one is the related joint determinant of a joint a is can be obtained from the determinant of a nothing but n minus 1

so where yeah is a is a is android matrix okay here is n cross n matrix okay yeah

so this also this is also not very difficult to difficult to prove okay

so yeah

so i think i think these are the more or less the properties which we will be using when we solve the problems related with determinant and matrices

so let's solve the question number one let m be three cross three matrix

so this one m times $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is equals to minus $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ okay then what will be the the sum of the diagonal entries of m

so let's solve this problem okay

so suppose m is given by this $m_{11} \ m_{12} \ m_{13} \ m_{21} \ m_{22} \ m_{23} \ m_{31} \ m_{32} \ m_{33}$ suppose m is given by this okay then we need need to find the value of m_{11} plus m_{22} plus m_{33} okay

so let's see how to how to do that all right

so we take the first equation first question says that m times $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is equals to minus $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

so this implies this implies if you multiply matrix with $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ then you will get here you get $m_{12} \ m_{22} \ m_{32}$ this is equals to minus $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ this implies m_{12} is minus 1 m_{22} is 2 m_{32} is 3 ok

so let's take the second equation which is m times $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is equals to $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and this implies if you multiply m with the $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ then you get m_{11} minus m_{12} .

m_{21} minus m_{22} m_{31} minus m_{32} this is equals to $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

so this implies m_{11} minus m_{12} is 1 implies m_{11} is equals to 1 plus m_{12} m_{12} is from here if you see minus 1

so it's 1 minus 1 0

so m_{11} is 0 now

so this is the first entry of a diagonal this is the second diagonal entry okay

so okay let's see another equation m_{21} minus m_{22} is equals to 1

so this implies m_{21} is equals to 1 plus m_{22}

so 1 plus m_{22} is 2 is equals to 3 okay and the last one is m_{31} minus m_{32} is equals to minus 1 all right this implies m_{31} is equals to minus 1 plus m_{32} what is m_{32} is 3 3 minus 1 is 2 okay

so now yeah

so these are the these are the values we have

so now let's go back to the because still we need to calculate m_{33} the last equation

so last equation is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is equals to $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

so this implies that m_{11} plus m_{12} plus m_{13} m_{21} plus m_{22} plus m_{23} m_{31} plus m_{32} plus m_{33} this is equals to 0 0 okay

so from previous slide we i think i know the value of m_{31} and m_{32} yes and m_{31} is 2 and m_{32} is 3 all right then let's take the this equation third one m_{31} plus m_{32} plus m_{33} is equals to 12

so m_{33} is nothing but 12 minus 2 and 3 one minus 3 two and what is m_{31} let me just see from the previous one and m_{31} is 2 and m_{32} is 3 2 plus 3

so 12 minus 2 minus 3 it is 7

so we get m_{33} as a 7 okay

so this means m_{11} plus m_{22} plus m_{33} is nothing but m_{11} one i remember correctly it was zero m_{22} was 2

so 0 plus 2 plus 7 is equals to 9 this is the final answer question let w not equals to 1 b a cube root of unity and S the set of sorry let me just write an end and S in the set of all non-singular matrices of the form $\begin{pmatrix} a & b & c \\ w & w & w \\ w & w & w \end{pmatrix}$ where each of a, b, c is either w or w^2 then what is the cardinality of this S okay

so now we get $a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33}$ okay
so this is equals to this is the determinant of b right $2 \ 3 \ 5 \ 9 \ 10 \ 12 \ 2$ to the power 12 into determinant of p and we know the determinant of p is 2

so determinant of q is equals 2 to the power 12 into determinant of p which is 2 equals to 2 to the power 13

so this is the final answer question let p be a 3 cross 3 matrix such that p transpose is equals to $2p + I$ where I is the three cross three identity matrix consider a column matrix which is also called the column vector x and $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ which is not equal to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ that is we are considering a non-zero vector x then which of the following is true okay

so the first option is $p \cdot x$ is equals to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ the second option is $p \cdot x$ is equals to x the third option is $p \cdot x$ is equals to $2x$ and whole one is $p \cdot x$ is equals to minus x okay

so let's solve this problem okay

so let's start with the what is what is given to us which is the p transpose is equals to $2p + I$ let's call this equation 1

so t transpose on both side that means p transpose let me just erase this v transpose of transpose this is equals to $2p + I$ transpose and this is nothing but p is equals to $2p$ transpose plus I transpose which is I itself let's call the equation 2.

so now what is p transpose minus p this is equals to 2 times p minus p transpose all right

so this implies that 3 times p transpose minus p is equals to zero and this zero is the zero matrix okay the matrix which has all the entries zero

so this implies p transpose is equals to p

so that is p is a symmetric matrix okay

so now given p transpose is equals to p then the from equation 1 if you see you see p is equals to $2p + I$ implies p is equals to minus I all right

so this is what we obtained okay now let's check let's check uh which options are two okay

so the first one is $p \cdot x$ is equals to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ this implies that p is minus I

so minus $I \cdot x$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and this implies this gives x as $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ okay but x is given to be non-zero

so this implies 1 is not true okay the second is $p \cdot x$ is equals to x the for $p \cdot x$ is $p \cdot x$ is equals to x and then this implies that $p \cdot x - p \cdot x$ is minus I

so $p \cdot x - p \cdot x$ is equals to x this implies that x is again $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and here $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the this is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ vector but x is nonzero right this big zeros I mean they denote the zero vector right

so so that means uh two is also not true okay similarly 3 is also not true because 3 will also lead 3 is this white only if x is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

so similarly 3 is not 2.

okay then the fourth one

so fourth one is that $p \cdot x$ is equals to minus x

so this is true because $p \cdot x$ is equal to minus x

so this is true because p is minus I

so we can say that for all x not equals to zero zero zero

so four is true

so out of four options only only the last one the fourth one is correct because other three are satisfied only if x is equals to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ but x is given that it's non-zero okay okay students

so i'll stop here now i'll solve some more problems related with matrices and determinant in the next lecture thank you you